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Erratum to "On Operations and Linear Extensions of Well Partially Ordered Sets"

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In this article, we give a counter-example to Lemma 12 of the article "On Operations and Linear Extensions of Well Partially Ordered Sets" by Maciej Malicki and Aleksander Rutkowski.

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1 Introduction

In this article, we give a counter-example to Lemma 12 of the article "On Operations and Linear Extensions of Well Partially Ordered Sets" by Maciej Malicki and Aleksander Rutkowski (Malicki and Rutkowski (2004)).

2 Definitions and notations

Definition 2.1 (Rank function)**.** *Each well-founded poset* P *admits an ordinal valued rank function* rank_P *defined inductively on its elements:* $\mathrm{rank}_P(a) = sup_{x<_P a}(\mathrm{rank}_P(x) + 1)$

Let $\mathcal{P} = \{P_t : t \in T\}$ be an ordered family of ordered sets, i.e. both P_t's and T are partially ordered (by $\leq_t$ and $\leq_T$ respectively). With no loss of generality, elements of $\mathcal{P}$ can be assumed to be pairwise disjoint. Let, for $a \in \bigcup_{t\in T} P_t$, $f(a)$ be that unique t such that $a \in P_t$.

Now, assume all elements of $\mathcal{P}$ to be well-founded and call, for $a \in \bigcup_{t\in T} P_t$, the *primitive rank* of a an ordinal $g(a) = \mathrm{rank}_{P_{f(a)}}(a)$. Define the following ranked order $<_R$ on $\bigcup_{t\in T} P_t$: $a <_R b$ if

- either $f(a) = f(b)$ and $a <_{f(a)} b$,
- or $f(a) < f(b)$ and $g(a) \leq g(b)$.

Call the union with that order the *ranked sum* and denote it $\mathcal{RP}$. Observe that $a \leq_R b$ implies $f(a) \leq_T f(b)$.

3 A counter example to Lemma 12

False lemma 3.1 (Lemma 12)**.** *Let both T and all the components P_t of $\mathcal{RP}$ be well-founded (hence $P = \mathcal{RP}$ is well-founded too). Then for each $a \in \bigcup_{t\in T} P_t$, $\mathrm{rank}_P(a) \leq \mathrm{rank}_T(f(a)) + \mathrm{rank}_{f(a)}(a)$.*

Counter-example:

> It is easy to construct an order $\mathcal{RP}$ with an element a such that $\mathrm{rank}_P(a) = \mathrm{rank}_T(f(a)) + \mathrm{rank}_{f(a)}(a) + 1$. Indeed consider $T = \{0, 1\}$, and $P_0 = P_1 = \omega + 1$ (ω is the first infinite ordinal). Let a be the maximum of P_1, and b be the maximum of P_0. Then $\mathrm{rank}_{\mathcal{RP}}(b) = \mathrm{rank}_{P_0}(b)$, hence $\mathrm{rank}_{\mathcal{RP}}(a) = \mathrm{rank}_{\mathcal{RP}}(b) + 1 = \omega + 1 > \mathrm{rank}_T(f(a)) + \mathrm{rank}_{f(a)}(a) = \mathrm{rank}_T(P_1) + \mathrm{rank}_{P_1}(a) = 1 + \omega = \omega$ (ordinal sum is not commutative and $1 + \omega \neq \omega + 1$).
>
> The problem in the proof is in the line $\mathrm{rank}_T(f(b)) + \mathrm{rank}_{f(b)}(b) + 1 \leq \mathrm{rank}_T(f(a)) + \mathrm{rank}_{f(a)}(a)$. It should be corrected to $\mathrm{rank}_T(f(b)) + 1 + \mathrm{rank}_{f(b)}(b) \leq \mathrm{rank}_T(f(a)) + \mathrm{rank}_{f(a)}(a)$, but then the proof by transfinite induction fails.
>
> You cannot correct the lemma by switching both ranks, i.e. $\mathrm{rank}_P(a) \leq \mathrm{rank}_{f(a)}(a) + \mathrm{rank}_T(f(a))$. Indeed then $T = \omega + 1$, and $P_0 = P_1 = ... = P_\omega = \{0, 1\}$ is a counter-example.

■

Lemma 3.2. *For any ordinal α, there is an order $\mathcal{RP}$ with an element a such that* $\mathrm{rank}_P(a) = \mathrm{rank}_T(f(a)) + \mathrm{rank}_{f(a)}(a) + \alpha$.

Proof: Consider $T = \alpha + 1$, and $P_0 = P_1 = ... = P_\alpha = \beta + 1$, where β is the first ordinal such that $\alpha + \beta = \beta$. Let a be the maximum of P_α, and b be the maximum of P_0. Then $\mathrm{rank}_{\mathcal{RP}}(b) = \mathrm{rank}_{P_0}(b)$, hence $\mathrm{rank}_{\mathcal{RP}}(a) = \mathrm{rank}_{\mathcal{RP}}(b) + \alpha = \beta + \alpha > \mathrm{rank}_T(f(a)) + \mathrm{rank}_{f(a)}(a) = \mathrm{rank}_T(P_\alpha) + \mathrm{rank}_{P_\alpha}(a) = \alpha + \beta = \beta$. ■

Lemma 3.3. *For any ordinal α, there is an order $\mathcal{RP}$ with an element a such that* $\mathrm{rank}_P(a) = \mathrm{rank}_{f(a)}(a) + \mathrm{rank}_T(f(a)) + \alpha$.

4 Conclusion

We sent an email to one of the authors on 2019/02/24 but we never had an answer. It is hard to tell if it is because of our lack of communication skills, or because our email or his answer was intercepted by spies.

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