First difference principle applied to modular/questionable-width, clique-width, and rank-width of binary structures

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Abstract

In this article, we study various widths/decompositions of binary structures under the light of first difference principle. We show the equality of modular-width and questionable-width, although questionable representation is more primitive/crude than modular/clan decomposition. Using first difference principle, we show that clique-decomposition of binary structures has a sequential equivalent, that we call clique-questionable representation. Last but not least, we give remarks on the true validity of rank-width of binary structures.

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1 Introduction

Apologies: We do science as a hobby, it is not our daily job and there is an impact on the quality of the bibliography. For an unpublished work we did in 2015, we started doing bibliographic search during 9 months, but all the gathered references were lost when a hacker erased all our files on our laptop. Since then, we chose to publish our ideas on arXiv and correct the bibliography afterwards. For preparing this article, we have read entirely the book "The Theory of 2-structures" by Ehrenfeucht et al. (1999), the thesis of Bui-Xuan (2008), and a few articles.

In this article, we study various width/decompositions of binary structures under the light of first difference principle. First difference principle is the principle at the heart of many mathematical objects:

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- lexicographic order, where it is used before any length consideration,
- hierarchic order, positional notation or positional numeral system, where it is used after length consideration, etc.

Thus it is at least 5000 years old. To our knowledge, it was named "Principe de première différence" by Sierpiński (1932). It was previously used by Hausdorff (1907) for universal orders. The idea is simply to define a mathematical object with a sequence of "smaller" (in most cases, finite) objects/items. Then two objects/sequences can be compared, linked by a relation resp., according to the order, relation resp., between the first different elements in the sequences. These objects defined by sequences are in turn elements of (a set) another mathematical object, and at least one mathematical property of these subobjects in these bigger object is faithfully preserved by these sequences of items under the first difference principle. Thus first difference principle is used for binary structures : orders, graphs, (labelled) 2-structures. To our knowledge, a "3-way" first difference principle (like we have 3-way merge for source code) is yet to be defined, if it is possible (it should be first differences principle).

After studying first difference principle for orders in Lyaudet (2018) and Lyaudet (2019), it was natural for us to consider that this principle could be used for binary structures by applying Zermelo's axiom three times. We illustrate this by showing how to convert a labelled properties graph model used in graph databases into a labelled 2-structure (see Ehrenfeucht et al. (1999)).

Theorem 1.1. *Testing if a logical formula using quantification over vertices or set of vertices is true on a labelled properties graph is equivalent to testing if a logical formula using quantification over vertices or set of vertices is true on a labelled 2 structure.*

Proof:

This proof is written with examples and lacks rigour. It is intentional: The result is easy, and this is the introduction.

Assume that you have enumerated all possible directed relations you have in some labelled properties graph; for example, you have vertices representing persons, cars, and addresses, and the following directed relations with properties:

- "is_mother_of" links a mother to its child with a property "age_at_birth" giving the age of the mother when she gave birth to this child,
- "is_father_of" links a father to its child with a property "age_at_birth" giving the age of the father when his wife gave birth to this child,
- "play_tennis_with" links two persons,
- "has_home_at" links a person to the address vertex corresponding to its home,
- "owns" links a person to its car with a property "buying_date" and a property "buying_location".

As a first step, you use Zermelo's axiom (in the infinite case) a first time to list all the combinations of a relation plus its properties that appear in the graph, like first combination is ("is_mother_of", {"age_at_birth": 31}), second combination is ("is_mother_of", {"age_at_birth": 24}), third combination is ("has_home_at", {}), etc.

For the sake of efficiency, the first step of the transformation should depend of the formula. For example, if you have a formula "there exists a person p such that, (there exists a car c such that, p owns c) and (for all car c', p owns c' implies p bought c' before 2020/01/01)", then you will first forget all edges corresponding to the relations "is_mother_of", "is_father_of", "has_home_at". Moreover you should also forget all properties of "owns" except "buying_date" ("buying_location" has no usefulness for the problem at hand). Thus you will enumerate only the combinations ("owns", {"buying_date": 2011/01/01}), ("owns", {"buying_date": 2012/01/01}). You have a ternary relation LinkedBy(x,y,comb) where x and y are vertices and comb is the integer/ordinal corresponding to a combination of a relation plus its properties. Assume that the combinations 1, 3, 4 are all the combinations of "owns" relation, and that the combinations 1, 3 are all the combinations of "owns" relation with buying date before 2020/01/01. So far, the formula becomes "there exists a person p such that, (there exists a car c such that, LinkedBy(p,c,1) or LinkedBy(p,c,3) or LinkedBy(p,c,4)) and (for all car c', LinkedBy(p,c',1) or LinkedBy(p,c',3) or LinkedBy(p,c',4) implies LinkedBy(p,c',1) or LinkedBy(p,c',3))".

Note that in these step and the two other steps coming next, we may use a disjonction of an infinite number of atomic subformulas, when the binary structure is infinite. We consider that the OR nodes/gates have unbounded fan-in. Thus the depth of the formula only increases by 1.

As a second step, you use Zermelo's axiom (in the infinite case) a second time to list all the set of combinations of a relation plus its properties that appear between two vertices in the graph with the same in-vertex and out-vertex, like first set of combinations is {("play_tennis_with", {})}, second set of combinations is {("play_tennis_with", {}), ("is_mother_of", {"age_at_birth": 31})}, third set of combinations is {("is_mother_of", {"age_at_birth": 24})}, fourth set of combinations is {} (no relations), etc. Any couple (ordered pair) of vertices has a unique corresponding set of combinations, and each set of combinations has at least one corresponding couple of vertices.

We kept the description of the combinations for clarity but at this step, combinations are already integers/ordinal numbers, like first set of combinations is {1}, second set of combinations is {1,5}, third set of combinations is {3}, fourth set of combinations is {} (no relations), etc. Thus, it is easy to find duplicate sets using lexicographic order and ordering combinations inside each set. You have a ternary relation AdjacencySemiType(x,y,s) where x and y are vertices and s is the integer/ordinal corresponding to a set of combination of a relation plus its properties.

Back at our example, assume that the set of combinations 2, 5, 11 are all the set of combinations containing "owns" relation, and that the set of combinations 2, 11 are all the set of combinations containing "owns" relation with buying date before 2020/01/01. So far, the formula becomes "there exists a person p such that, (there exists a car c such that, AdjacencySemiType(p,c,2) or AdjacencySemiType(p,c,5) or AdjacencySemiType(p,c,11)) and (for all car c', AdjacencySemiType(p,c',2) or AdjacencySemiType(p,c',5) or AdjacencySemiType(p,c',11) implies AdjacencySemiType(p,c',2) or AdjacencySemiType(p,c',11))".

After using two times Zermelo's axiom, we already have a labelled 2-structure and a wealth of results starts to apply. Our example is not realistic because there should be probably much more AdjacencySemiTypes than combinations.

Now, we need to apply Zermelo's axiom a third time in order to use first difference principle. Indeed, the adjacency type of the couple (x,y) is the couple of the two adjacency semi-types from x to y and from y to x. Again, we may fear a quadratic increase in the number of adjacency types compared to the number of adjacency semi-types, when enumerating them. Our formula may become "there exists a person p such that, (there exists a car c such that, AdjacencyType(p,c,3) or AdjacencyType(p,c,14) or AdjacencyType(p,c,111)) and (for all car c', AdjacencyType(p,c',3) or AdjacencyType(p,c',14) or AdjacencyType(p,c',111) implies AdjacencyType(p,c',14) or AdjacencyType(p,c',111))". We now have what is called a reversible labelled 2-structure. It is reversible because now the function that gives the adjacency type from x to y has the property that whenever two couples of vertices (x,y) and (x',y') have the same adjacency type, then the two couples of vertices (y,x) and (y',x') have the same adjacency type.

Clearly, the formula has only been modified by substituting a disjunction of a (probably high) number of atomic subformulas (AdjacencyType(x,y,t)) to a single atomic subformula (e.g. p owns c') in the original formula. Its main structure remained the same.

We note that LinkedBy must be a ternary relation, but AdjacencyType and AdjacencySemiTypes are equivalently ternary relations or binary functions. All three are directed.

Last but no least, it is easy to see how to deal with multiple edges/arcs. On this respect, if we have k arcs from x to y like "has_won_a_tennis_match_against", we keep only one arc "has_won_a_tennis_match_against" and we had a special property "multiplicity" of value k on this arc. If we had previous properties like ("has_won_a_tennis_match_against", {"date": 2020/01/01}), ("has_won_a_tennis_match_against", {"date": 2020/01/01}), and ("has_won_a_tennis_match_against", {"date": 2020/01/03}), we cannot merge the last with the two previous; we have ("has_won_a_tennis_match_against", {"date": 2020/01/01, "multiplicity": 2}), ("has_won_a_tennis_match_against", {"date": 2020/01/03, "multiplicity": 1}). It does not change the three steps since we start from the combinations of a relation plus its properties, instead of bare relations. ■

If you think of labelled 2-structures as matrices, reversible labelled 2-structures are somehow "weakly symmetric" matrices. We bought the book by Ehrenfeucht et al. (1999) wanting to know if it was already known that "weakly symmetric" matrices/2-structures are sufficient to study. In this book, they justify reversibility relatively to clans but do not say a word of logical formulas that are marginally modified. The result for clans unchanged under reversibility can be seen as a corollary of our theorem, since clans can be defined using a formula using quantification over vertices only. Thus, it is clear that for many formulas and binary structures, we can use the first difference principle and reversible labelled 2-structures. We wrote this theorem thinking it was the most missing point in the introduction of Ehrenfeucht et al. (1999), in order to justify the study subject.

Section 2 contains the proof of equivalence between modular-width and questionable-width. In section 3, we show how clique-width belongs to the scope of first difference principle. Section 4 concludes with remarks on rank-width of (reversible) (labelled) 2-structures.

2 Modular-width and questionable-width

We want to apologize to our readers of our previous articles on first difference principle. Probably, many of them already knew modular and clan decompositions, and they immediately have seen that questionable-width was equivalent to modular/clan-width. We did not work on modular decomposition previously and what we remembered of talks about it was erroneously that you would decompose a graph/2-structure when it was either disconnected (parallel node) or its complement was disconnected (series node), stopping when both are connected. Our further readings corrected this mistake and we have seen that the modular/clan decomposition goes on under primitive nodes. At this point, it was immediate to see that questionable-width was equivalent. See Ehrenfeucht et al. (1999), and Habib and Paul (2010) for a survey on algorithmic modular/clan-decomposition.

Consider a binary signature $\mathcal{S}$ of binary relations and functions. This is equivalent to have a set of arcs with label and properties, as in labelled properties graph model used in graph databases. Given a set S, an $(\mathcal{S}, S, k, l)$*-mapping-run* is an (ordinal-indexed) sequence $(S_i)_{i\in l}$ of length l of $\mathcal{S}$-structures of cardinality at most k, together with a sequence of mappings from S to the domains of S_i structures.

Definition 2.1 (Questionable representation)**.** *A* questionable representation *of a reversible labelled 2-structure S is a $(\mathcal{A}, S, k, l)$-mapping-run, for some k and l, where $\mathcal{A}$ is its set of adjacency types, such that first difference principle applied to this mapping-run yields the adjacencies of S. k is its width and l is its length. The* questionable-width *of S is the minimum of the widths of its questionable representations.*

The modular/clan-width is the maximum of the cardinals of the domains of quotient graphs/2-structures that appear in primitive nodes of the modular/clan decomposition of a given graph/2-structure. We fix the following convention that the modular/clan-width of a graph/2-structure without primitive nodes in his modular/clan decomposition is exactly two.

Under this convention, we have the following result.

Theorem 2.2. *The questionable-width of a reversible labelled 2-structure is equal to its clan-width.*

Proposition 2.3. *The questionable-width of a reversible labelled 2-structure S is at most its clan-width.*

Proof:

We process the nodes of the clan-decomposition from the root to its leaves, the order of the nodes is unimportant as long as the set of treated nodes is connected at any step. At any step, we lengthen a sequence of hopefully small 2-structures S_i, such that any vertex of the 2-structure S is associated to exactly one vertex in each S_i. Our induction hypothesis is that if a node has been treated, then all vertices of the 2-structure that belongs to distinct branches below this node have their adjacency types that are already set by first difference principle. Thus we obtain a questionable representation at the final step.

If the current node to be treated is an α-complete node where α is a symmetric adjacency type, or an α-linear node where α is an asymmetric adjacency type, we add at the end of our current partial questionable representation a sequence of 2-structures of size 2, their two vertices x_j^0 and x_j^1 having α adjacency type on the couple (x_j^0, x_j^1). If the number of nodes under the current node is k, we need $\lceil \lg(k) \rceil$ such small 2-structures, and we associate to each vertex y of S under the current node to x_j^0 or x_j^1 using the binary notation of the ordinal finite number corresponding to the subnode containing y. (The binary notations must be padded on the left with 0s to have the same length.) All other vertices of S are associated to $x_j^0, 1 \leq j \leq \lceil \lg(k) \rceil$.

If the current node to be treated is a primitive node, we add at the end of our current partial questionable representation a 2-structures isomorphic to the quotient 2-structure at the primitive node. We associate to each vertex y of S under the current node the vertex of its maximal prime clan in the quotient 2-structure. All other vertices of S are associated to the same arbitrary vertex in the quotient 2-structure.

■

Proposition 2.4. *The questionable-width of a reversible labelled 2-structure is at least its clan-width.*

Proof:

Assume for a contradiction that there is a primitive node separating vertices x and y, and that the 2-structure in the questionable representation, where the first difference between x and y occurs, has size less than the cardinal of the quotient 2-structure at the primitive node. We take such a pair of vertices x and y with minimum first difference index. Clearly, without loss of generality, there must be some vertex z that is in a subbranch below the primitive node; this subbranch is distinct of the subbranches of x and y; and nevertheless x and z have the same image in the first difference 2-structure. But then since the preimage of the image of x (and z) restricted to the vertices below the primitive node is a clan of the substructure induced by these vertices, it contradicts the fact that the quotient 2-structure was primitive.

■

The previous results solve the problem of optimizing the width of a questionable representation. But there remains some open-problems :

Open problem 2.5. *Optimize the length of a questionable representation under optimality of the width or a relaxed assumption on the width. (Without width constraint this is obviously one.) Optimize the area (sum of the cardinals of the domains of the labelled 2-structures in the questionable representation) or the approximated area (length times width) of a questionable representation, with or without width constraint.*

Open problem 2.6. *Some graphs admits questionable representations with logarithmic length, area, and approximated area, under optimal width constraint. Find equivalent characterizations of such graphs. (All labelled 2-structures admits questionable representations with linear length, area, and approximated area, under optimal width constraint.)*

Acccording to Ehrenfeucht et al. (1999), and Habib and Paul (2010), the origin of modular/clan decomposition can be traced back to Gallai (1967). Putting it in the scope of first difference principle, we go a few milleniums before but graphs and 2-structures were not studied yet. It would be wrong to say that modular/clan decomposition can be traced back to a few milleniums before, but we can say that a simple principle was already known that could lead to it with some work.

3 Clique-width and first difference principle

A few years ago, we did not know much about 2-structures, and we generalized clique-width with weighted graphs, the weights being over some field (see Flarup and Lyaudet (2008)). When we read the book by Courcelle and Engelfriet (2012), it was a surprise to see their generalization to labelled edge graphs using separate edge families. However the two generalizations are equivalent, may it be using ordinal weights for infinite graphs, or rational numbers weights for finite binary structures. This is a consequence of the theorem in introduction and of the following fact: with clique-width, you can add all the edges between two vertices as soon as the two paths from the leaves corresponding to two vertices to the root of the clique-decomposition joins the same node in the decomposition; hence vertices coming from the same branch with distinct adjacency types to a vertex of the other branch must have distinct clique-labels.

In order to find a sequential/first difference principle equivalent to clique-width and clique decomposition, we need to introduce nyf-extended labelled 2-structures. nyf is a constant special adjacency type meaning "not yet fixed". Only reversible labelled 2-structures appearing in decompositions may use this special adjacency type. Its meaning is obvious, when the first difference between two vertices that occurs in a nyf-extended labelled 2-structures, has the nyf-adjacency type between the images of the two vertices then their adjacency type must be given by another difference later. Thus, nyf-extended first difference principle is that the adjacency type is given by first difference where adjacency type is not nyf.

There is a dual constant to nyf-extended labelled 2-structures. This is alf-extended labelled 2-structures. alf is a constant special adjacency type meaning "already fixed". Only reversible labelled 2-structures to be decomposed may use this special adjacency type. This is what is used to further decompose some reversible labelled 2-structure in tree-questionable-width. (We introduced tree-questionable-width in Lyaudet (2019).)

It is somewhat linked to bi-modular decomposition in this setting Fouquet et al. (2004). We shall use it in a future article on classes of bounded balanced tree-questionable-width, but not in this article.

A nyf-connected-component, resp. nyf-clique, in a nyf-extended labelled 2-structures is simply a connected component, resp. a clique, in the underlying undirected simple graph where nyf-adjacency corresponds to edges and everything else correspond to a non-edge.

Definition 3.1 (Clique-questionable representation)**.** *A* clique-questionable representation *of a reversible labelled 2-structure S is a* $\mathrm{nyf}-(\mathcal{A}, S, k, l)$*-mapping-run, for some k and l, where $\mathcal{A}$ is its set of adjacency types, such that:*

(i) nyf*-extended first difference principle applied to this mapping-run yields the adjacencies of S;*

(ii) the nyf*-extended reversible labelled 2-structures must have the property that there are exactly two* nyf*-connected-components, and that these* nyf*-connected-components are* nyf*-cliques;*

(iii) two vertices S that have a nyf*-difference must have only* nyf*-differences until they have a non-*nyf*-difference; they cannot be mapped to the same vertex again until some non-*nyf*-difference occurs.*

k is its width and l is its length. The clique-questionable-width *of S is the minimum of the widths of its clique-questionable representations.*

Theorem 3.2. *The clique-questionable-width of a reversible labelled 2-structure is between its clique-width and two times its clique-width.*

Proposition 3.3. *The clique-questionable-width of a reversible labelled 2-structure S is at most two times its clique-width.*

Proof:

Without loss of generality, we may assume that all edges are added in the clique-decomposition as soon is possible, using the compact clique-algebra (Courcelle and Engelfriet (2012)). Moreover, at the cost of doubled clique-width, we assume that when two branches joins in the clique decomposition, their port labels are disjoint. We process the (compact-)nodes of the clique-decomposition from the root to its leaves, the order of the nodes is unimportant as long as the set of treated nodes is connected at any step. At each step, we lengthen a sequence of hopefully small nyf-extended labelled 2-structures S_i, such that any vertex of the 2-structure S is associated to exactly one vertex in each S_i, respecting condition (iii) above. Moreover, each S_i is the union of two nyf-cliques. Our induction hypothesis is that if a node has been treated, then all vertices of the 2-structure that belongs to distinct branches below this node have their adjacency types that are already set by nyf-extended first difference principle (condition (i')), and conditions (ii) and (iii) are respected. Thus we obtain a clique-questionable representation at the final step.

Since the compact clique-algebra yields a single type of nodes (no need to handle the leaves corresponding to isolated vertices), the induction is straight-forward.

We add a nyf-extended labelled 2-structures S_i made of two nyf-cliques, the first nyf-clique having as much vertices as port labels in the left branch, the second nyf-clique having as much vertices as port labels in all the clique-decomposition (condition (ii) is satisfied). The mapping of vertices from S to S_i associates all vertices below the current node to the vertex corresponding to its port label (under the current node) in the nyf-clique corresponding to its branch; other vertices are associated to the vertex corresponding to its port label (under the last corresponding node treated) in the nyf-clique of S_i having all port labels. The adjacency types in S_i between the two nyf-cliques are the ones corresponding to the node of the clique-decomposition with the additional no-edge default adjacency type explicitly set, when the compact-node adds nothing between two port labels sets of vertices. Clearly condition (i') is satisfied. To see that condition (iii) is satisfied, we observe that:

- for all vertices below the current node, condition (iii) is true between two vertices in the same branch below the current node, because it was true previously, and no renaming operation may give distinct port labels if they had the same port label below the node; two vertices in distinct branch have now distinct port labels and an adjacency type set;
- for all other vertices, condition (iii) was satisfied previously, and nothing changed with the previous step with respect to being mapped to the vertex or not;
- for a vertex below the current node and a vertex not below, condition (iii) was satisfied previously, and they have already a non-nyf difference, hence it will never be unsatisfied in the future.

Q.E.D. ■

Proposition 3.4. *The clique-questionable-width of a reversible labelled 2-structure is at least its clique-width.*

Proof:

Given a clique-questionable representation, we construct a (compact)-clique decomposition of same width in a top-down manner. First, using a number of port labels equal to the width of the clique-questionable representation, we assign a distinct port label to each vertex of S_i, for all i; this choice is completely arbitrary as long as two distinct vertices of the same S_i have distinct port labels. At each step, we maintain a set of subsets of S, $\{X_j\}$, that have already been splitted, all adjacency types between distinct $X_j, X_{j'}$ being already fixed both by the beginning of clique-questionable representation and the in-construction (compact-)clique-decomposition. Each X_j corresponds to a leaf of the in-construction (compact-)clique-decomposition. Before first step, $\{X_j\} = \{S\}$. For each (step) S_i item in the clique-questionable representation, consider the substructure (substep) $S_{i,j}$ of S_i restricted to the image of X_j. If $S_{i,j}$ is included in one of the two nyf-cliques, then this substep is finished. Otherwise $S_{i,j}$ is made of two parts $S_{i,j,0}$ and $S_{i,j,1}$ corresponding to the two nyf-cliques. We replace the leaf corresponding to X_j with a node that renames the port labels of the vertices in $S_{i,j}$ into the port labels of X_j vertices at the substep X_j was created (we can do so with respect to distinct vertices in the same $S_{i,j,b}$ thanks to condition (iii)); under this node we add adding adjacency type operations/nodes to reflect $S_{i,j}$ (this is easy since all port labels are distinct); last we had a disjoint union node separating two leaves corresponding to $X_{j,0}, X_{j,1}$ the domains of $S_{i,j,0}, S_{i,j,1}$. (The nodes created at a substep can trivially be grouped into a compact-node.) It is trivial to see that the clique-decomposition obtained constructs the same labelled 2-structure as the clique-questionable representation, and does not use more port labels than its width. ■

Proposition 3.3 can not be improved unless Proposition 2.105 (4) in Courcelle and Engelfriet (2012) can be improved. (We do not think it can.) First difference principle, as we use it, is totally "adjacency type impartial", it does not make any difference between some default adjacency type such as no-edge and other adjacency types. It has the advantage that all results apply under any injective mapping from a set of adjacency types to another. Hence, the two previous definitions also define the questionable-width and the clique-questionable-width of a 2-structure instead of a labelled 2-structure. It would be possible to shrink (clique-)questionable representations in some cases, with the convention that, when no two differences occurs, a default adjacency type is given. Non-edges gives an example of such a symmetric default adjacency type. Lexicographic order gives an example of such an asymmetric default adjacency type, using the length to determine the orientation. We do not study further this topic yet because we think that as the saying goes "Explicit is better than implicit", and "Premature optimization is the root of all evil" (Knuth). Further research may prove us wrong. Courcelle generalized clique-decompositions to countable graphs/labelled 2-structures. nyf-extended first difference principle and clique-questionable representation give us an alternative for reversible (labelled) 2-structures of any cardinality.

In light of the introductory theorem and the "good behaviour" of clique-width and monadic second order logic with vertex quantification, it was an easy guess that first difference principle may have a link with clique-width.

Déjà vu:

Open problem 3.5. *Optimize the length of a clique-questionable representation under optimality of the width or a relaxed assumption on the width. (Without width constraint this is obviously one.) Optimize the area (sum of the cardinals of the domains of the labelled 2-structures in the clique-questionable representation) or the approximated area (length times width) of a clique-questionable representation, with or without width constraint.*

Open problem 3.6. *Some graphs admits clique-questionable representations with logarithmic length, area, and approximated area, under optimal width constraint. Find equivalent characterizations of such graphs. (All labelled 2-structures admits questionable representations with linear length, area, and approximated area, under optimal width constraint.)*

4 Rank-width of labelled 2-structures

In 2008, we were working to extend the results of Barvinok (1996) and Flarup and Lyaudet (2008). Barvinok showed that if the underlying matrix has bounded rank, both the permanent and the hamiltonian polynomials can be evaluated in polynomial time; we showed a similar result for bounded weighted clique-width matrices. We hoped to show that the permanent of a bounded rank-width matrix can be computed in polynomial time. Our opinion at that time was that rank-width was a kind of algebraic trick to "compress" clique-width. We did not know whether rank-width and clique-width are still equivalent over any arbitrary field. We talked about this subject with Frédéric Mazoit, who talked about it to Michaël Rao, who gave us the following example in 2010: Consider a finite set of vertices, each associated to a distinct strictly positive integer, giving the matrix over $\mathbb{Q}$ where the edge weight between vertices i and j is $i \times j$. This matrix has rank 1, hence has rank-width 1. But it should have clique-width around $n/2$, where n is the number of vertices. We can modify this example to guaranty that all edges are distinct: Consider a finite set of vertices, each associated to a distinct prime number p_i (We have a symmetric 2-structure). Or even that all arcs are distinct: cell i, j has weight $p_i^2 \times p_j$ (We have an asymmetric reversible 2-structure).

After that, our opinion on rank-width was that we should not be talking about equivalence between rank-width and other widths with too much confidence because rank-width lives in his own realm. It is able to capture "too predictable chaos" when all adjacencies are distinct.

At the light of first difference principle and "adjacency type impartiality", we think we should not focus on the algebraic meaning of rank-width. Instead we should feel free to compute the rank-width of any (labelled) 2-structure with an arbitrary injective mapping from its adjacency types to values in some arbitrary field. It yields two new widths of (labelled) 2-structures: maximum rank-width under injective mapping, and minimum rank-width under injective mapping. We have the corresponding open-problems:

Open problem 4.1. *For (finite) (labelled) 2-structures, what are the fields on which the maximum rank(-width) under injective mapping is obtained. Is it true for finite 2-structures that an injective mapping maximizing rank(-width) always exists over $\mathbb{Q}$? Find algorithms in the finite case.*

Open problem 4.2. *For (finite) (labelled) 2-structures, what are the fields on which the minimum rank(-width) under injective mapping is obtained. Is it true for finite 2-structures that an injective mapping minimizing rank(-width) always exists over the smallest prime field of cardinality at least the number of adjacency types of the 2-structure? Find algorithms in the finite case.*

Open problem 4.3. *Is it true that clique-width of 2-structures is closely equivalent to their maximum rank-width under injective mapping?*

Open problem 4.4. *Rank-width is able to capture "too predictable chaos", but minimum rank-width under injective mapping do capture "too predictable chaos". What kind of logic is suited for that? For example, it is clear that first order model checking is* *trivial on a 2-structure where all adjacency types are distinct, at least if the formula has no disjunction between adjacency types possible for some couple of quantified vertices in it.*

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References

A. Barvinok. Two algorithmic results for the traveling salesman problem. *Mathematics of Operations Research*, 21:65–84, 1996.

B. Bui-Xuan, M. Habib, V. Limouzy, and F. de Montgolfier. Homogeneity vs. adjacency: Generalizing some graph decomposition algorithms. In F. V. Fomin, editor, *Graph-Theoretic Concepts in Computer Science, 32nd International Workshop, WG 2006, Bergen, Norway, June 22-24, 2006, Revised Papers*, volume 4271 of *Lecture Notes in Computer Science*, pages 278–288. Springer, 2006. doi: 10.1007/11917496_25. URL https://doi.org/10.1007/11917496_25.

B.-M. Bui-Xuan. *Tree-Representation of Set Families in Graph Decompositions and Efficient Algorithms*. PhD thesis, Université Montpellier II, september 2008.

G. Cantor. Beiträge zur Begründung der transfiniten Mengenlehre. *Math. Ann.*, 46: 481–512, 1895.

B. Courcelle and J. Engelfriet. *Graph Structure and Monadic Second-Order Logic - A Language-Theoretic Approach*, volume 138 of *Encyclopedia of mathematics and its applications*. Cambridge University Press, 2012. ISBN 978-0-521-89833-1. URL http://www.cambridge.org/fr/knowledge/isbn/item5758776/?site_locale=fr_FR.

A. Ehrenfeucht, T. Harju, and G. Rozenberg. *The Theory of 2-Structures - A Framework for Decomposition and Transformation of Graphs*. World Scientific Publishing Co. Pte. Ltd., 1999. ISBN 981-02-4042-2.

U. Flarup and L. Lyaudet. On the expressive power of permanents and perfect matchings of matrices of bounded pathwidth/cliquewidth (extended abstract). In *CSR 2008, Computer Science in Russia*, volume 5010 of *Lecture Notes in Computer Science*, pages 180–193. Springer Verlag, 2008.

J. Fouquet, M. Habib, F. de Montgolfier, and J. Vanherpe. Bimodular decomposition of bipartite graphs. In J. Hromkovic, M. Nagl, and B. Westfechtel, editors, *Graph-Theoretic Concepts in Computer Science, 30th International Workshop,WG 2004, Bad Honnef, Germany, June 21-23, 2004, Revised Papers*, volume 3353 of *Lecture Notes in Computer Science*, pages 117–128. Springer, 2004.

doi: 10.1007/978-3-540-30559-0_10. URL https://doi.org/10.1007/978-3-540-30559-0_10.

T. Gallai. Transitiv orientierbare graphen. *Acta Mathematica Hungarica*, 18:25–66, 1967.

M. Habib and C. Paul. A survey of the algorithmic aspects of modular decomposition. *Comput. Sci. Rev.*, 4(1):41–59, 2010. doi: 10.1016/j.cosrev.2010.01.001. URL https://doi.org/10.1016/j.cosrev.2010.01.001.

F. Hausdorff. Untersuchungen über Ordnungstypen V. *Ber. über die Verhandlungen der Königl. Sächs. Ges. der Wiss. zu Leipzig. Math.-phys. Klasse*, 59:105–159, 1907.

L. Lyaudet. A class of orders with linear? time sorting algorithm. *CoRR*, abs/1809.00954, 2018. URL http://arxiv.org/abs/1809.00954.

L. Lyaudet. On finite width questionable representations of orders. *CoRR*, abs/1903.02028, 2019. URL http://arxiv.org/abs/1903.02028.

W. Sierpiński. Généralisation d'un théorème de Cantor concernant les ensembles ordonnés dénombrables. *Fundamenta Mathematicae*, 18:280–284, 1932.