On level-induced suborders

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Abstract

In this article, we characterize orders that are level-induced suborders anytime they are induced suborders of a superorder. We also characterize orders that are consecutive level-induced suborders anytime they are level-induced suborders of a superorder. Thus characterizing orders that are consecutive level-induced suborders anytime they are induced suborders of a superorder.

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Keywords : orders, always level-induced orders, ali orders, naturally consecutive level-induced orders, nacli orders, (directed) cographs, transitive series parallel graphs, interval orders, series parallel orders, series parallel interval orders, semi-orders, unit interval orders, series parallel unit interval orders, 1-weak orders

1 Introduction

Apologies: We do science as a hobby, it is not our daily job and there is an impact on the quality of the bibliography. For an unpublished work we did in 2015, we started doing bibliographic search during 9 months, but all the gathered references were lost when a hacker erased all our files on our laptop. Since then, we chose to publish our ideas on arXiv and correct the bibliography afterwards. For this article, we found no prior work defining kinds of induced suborders with constraints on their levels relatively to those of the superorder. Our search was in English and French scientific literature, and since the topic of order theory is ancient and vast, we may have missed early references in other languages. If you do know an early reference, please be kind enough to email/correct us.

This article study Open problem 5.16 in Lyaudet (2019). "Characterize finite orders that are induced suborders of any well-founded order if and only if they are (consecutive) level-induced suborders of this well-founded order. Examples: chains, antichains of size 1 and 2. Counter-examples: antichains of size at least 3."

Section 2 contains most of the definitions and notations used in this article. In section 3, we characterize orders that cannot be induced suborders without being level-induced suborders. Section 4 characterize orders that cannot be level-induced suborders without being consecutive level-induced suborders. In section 5, we give algorithms to find (level-)induced suborders of the previously defined classes.

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2 Definitions and notations

Throughout this article, we use the following definitions and notations. O will be reserved for asymptotic growth of functions. Thus P denotes an order (it may be either a partial, or a total/linear order), in particular $P^{0,1}$ denotes the binary total order where $0 < 1$. We denote $\mathrm{Domain}(P)$, the domain of the order P (for example, $\mathrm{Domain}(P^{0,1}) = \{0,1\}$). We write $x < y$, and $x > y$ as usual to express the order between two elements; we also write $x \sim y$ when two elements are incomparable in the partial order considered. We denote $\mathrm{OrderFunction}(P)$, the order function of the order P defined from $\mathrm{Domain}(P)^2$ to $\{=, \sim, <, >\}$ (for example, $\mathrm{OrderFunction}(P^{0,1}) = \{((0,0),=),\ ((0,1),<),\ ((1,0),>),\ ((1,1),=)\}$).

We denote $\mathrm{Inv}(P)$, the inverse/reverse order of P; for example, $\mathrm{Inv}(P^{0,1}) = P^{1,0}$ is the order on 0 and 1 where $1 < 0$.

Definition 2.1 (Maximum chain, height)**.** *Let P be an order, a chain of P is* maximum *if it is maximal and no other chain of P has greater cardinality. The cardinal of a maximum chain is the* height *of P, denoted* $\mathrm{Height}(P)$*. When P is well-founded*[1]*, we redefine a maximum chain to be one such that the corresponding ordinal is maximum; and we redefine its height to be the ordinal corresponding to its maximum chains. Thus in this case* $\mathrm{Height}(P)$ *denotes an ordinal.*

Note that an infinite order may have no maximum chain, but it always have at least one maximal chain. When there is no maximum chain, $\mathrm{Height}(P)$ is defined as the supremum cardinal/ordinal of the cardinals/ordinals corresponding to maximal chains.

In a well-founded order P, the level decomposition of P is the function $\mathrm{Level}_P : \mathrm{Domain}(P) \to \mathrm{Height}(P)$ ($\mathrm{Height}(P)$ is an arbitrary ordinal.) such that $\forall x \in \mathrm{Domain}(P), \mathrm{Level}_P(x) = \mathrm{Supremum}(\mathrm{Level}_P(y)+1$ such that $y < x, y \in \mathrm{Domain}(P))$. (Of course, this supremum is 0 if no element is below x.) We define the *level-width* of P as the supremum of the cardinals of the levels of P.

We consider the following kinds of suborders:

- An induced suborder P' of an order P is such that $\mathrm{Domain}(P') \subseteq \mathrm{Domain}(P)$, and $\forall x, y \in \mathrm{Domain}(P'), \mathrm{OrderFunction}(P')(x,y) = \mathrm{OrderFunction}(P)(x,y)$.
- A *level-induced suborder* P' of a well-founded order P is such that $\mathrm{Domain}(P') \subseteq \mathrm{Domain}(P), \forall x, y \in \mathrm{Domain}(P'), \mathrm{OrderFunction}(P')(x,y) = \mathrm{OrderFunction}(P)(x,y)$, and $\forall x, y \in \mathrm{Domain}(P'), \mathrm{Level}_{P'}(x) = \mathrm{Level}_{P'}(y) \Leftrightarrow \mathrm{Level}_P(x) = \mathrm{Level}_P(y)$. (Note that we could also define two other kinds of level-induced suborders with $\forall x, y \in \mathrm{Domain}(P'), \mathrm{Level}_{P'}(x) = \mathrm{Level}_{P'}(y) \Rightarrow \mathrm{Level}_P(x) = \mathrm{Level}_P(y)$ $\forall x, y \in \mathrm{Domain}(P'), \mathrm{Level}_{P'}(x) = \mathrm{Level}_{P'}(y) \Leftarrow \mathrm{Level}_P(x) = \mathrm{Level}_P(y)$. There is a simple proof by transfinite induction on the levels of P' showing that $\forall x, y \in \mathrm{Domain}(P'), \mathrm{Level}_{P'}(x) = \mathrm{Level}_{P'}(y) \Rightarrow \mathrm{Level}_P(x) = \mathrm{Level}_P(y)$ implies $\forall x, y \in \mathrm{Domain}(P'), \mathrm{Level}_{P'}(x) = \mathrm{Level}_{P'}(y) \Leftarrow \mathrm{Level}_P(x) = \mathrm{Level}_P(y)$. Moreover, the same proof shows that $\forall x, y \in \mathrm{Domain}(P'), \mathrm{Level}_{P'}(x) < \mathrm{Level}_{P'}(y) \Leftrightarrow \mathrm{Level}_P(x) < \mathrm{Level}_P(y)$. Thus only two kinds of level-induced

[1] Some authors also say Noetherian. In both cases, it means that there is no strictly decreasing infinite sequence.

suborders exists $\forall x, y \in \text{Domain}(P'), \text{Level}_{P'}(x) = \text{Level}_{P'}(y) (\Leftrightarrow$ or $\Rightarrow$ $) \text{Level}_P(x) = \text{Level}_P(y)$, and $\forall x, y \in \text{Domain}(P'), \text{Level}_{P'}(x) = \text{Level}_{P'}(y) \Leftarrow \text{Level}_P(x) = \text{Level}_P(y)$.)

- A *consecutive level-induced suborder* P' of a well-founded order P is such that $\text{Domain}(P') \subseteq \text{Domain}(P), \forall x, y \in \text{Domain}(P'), \text{OrderFunction}(P')(x, y) = \text{OrderFunction}(P)(x, y)$, $\forall x, y \in \text{Domain}(P'), \text{Level}_{P'}(x) = \text{Level}_{P'}(y) \Leftrightarrow \text{Level}_P(x) = \text{Level}_P(y)$, and $\forall x, y \in \text{Domain}(P'), \text{Level}_{P'}(x)+1 = \text{Level}_{P'}(y) \Leftrightarrow \text{Level}_P(x) + 1 = \text{Level}_P(y)$.

3 Orders that are always level-induced suborders

In this section, we assume that a given well-founded order P' is an induced suborder of a well-founded order P. We study necessary and sufficient conditions on P' to have that P' is a level-induced suborder of P.

Definition 3.1 (ali orders)**.** *An* ali order P' *is a well-founded order such that whenever P' is isomorphic to an induced suborder of a well-founded order P, then P' is also isomorphic to a level-induced suborder of P.*

We first observe that :

Lemma 3.2. *Any well-founded order P' is an induced suborder of a well-founded order P of level-width 2. Moreover, P has no level-induced suborder isomorphic to $O_{obs1} = (\{a, b, c, d\}, \{a < b, c < d\}) \equiv \text{Inv}(O_{obs1})$.*

Proof:

We use a well-founded chain to lift each element of $\text{Domain}(P')$ to a separate level. By Zermelo's axiom, there is a bijection f between some ordinal α and $\text{Domain}(P')$, such that $\forall x, y \in \text{Domain}(P'), \text{Level}_{P'}(x) < \text{Level}_{P'}(y) \Rightarrow f(x) < f(y)$. Assume, without loss of generality, that $\text{Domain}(P') \cap \alpha = \emptyset$.

Let $\text{Domain}(P) = \text{Domain}(P') \sqcup \alpha$:

- $\forall x, y \in \text{Domain}(P'), \text{OrderFunction}(P)(x, y) = \text{OrderFunction}(P')(x, y)$,
- $\forall x, y \in \alpha, \text{OrderFunction}(P)(x, y) = \text{OrderFunction}(\alpha)(x, y)$,
- $\forall x \in \alpha, \forall y \in \text{Domain}(P'), \text{OrderFunction}(P)(x, y) =$ ' $<$ ' if $x < f^{-1}(y)$, ' $\sim$ ' otherwise .

Clearly, P has all the claimed properties. ■

Corollary 3.3. *An ali order has level-width at most 2 and no level-induced suborder isomorphic to $O_{obs1} = (\{a, b, c, d\}, \{a < b, c < d\}) \equiv \text{Inv}(O_{obs1})$.*

Lemma 3.4. *No ali order has a level of size 2 except maybe the first.*

Proof:

Assume for a contradiction that P' is an ali order with two elements x, y such that $\mathrm{Level}_{P'}(x) = \mathrm{Level}_{P'}(y) > 0$. Take x, y such that their level is minimum. By the previous corollary, we must have an element $z, z < x, z < y$. (By transitivity, it is trivial to see that such a z exists in all previous levels, since only the first level may have two elements.) Thus we have a level-induced suborder isomorphic to $O_{obs2} = (\{a, b, c\}, \{a < b, a < c\})$.

We now show how to remove all such level-induced suborders for any well-founded order. Again by Zermelo's axiom, there is a bijection f between some ordinal α and $\mathrm{Domain}(P')$, such that $\forall x, y \in \mathrm{Domain}(P'), \mathrm{Level}_{P'}(x) < \mathrm{Level}_{P'}(y) \Rightarrow f(x) < f(y)$. This time, we add a distinct chain for each element of $\mathrm{Domain}(P')$. Let $\mathrm{DisjointCopy}(i)$ be a chain isomorphic to the ordinal i such that its elements are assumed to be distinct from all other elements considered in the following formula: $\mathrm{Domain}(P) = \mathrm{Domain}(P') \sqcup (\bigsqcup_{i \in \alpha} \mathrm{DisjointCopy}(i))$.

- $\forall x, y \in \mathrm{Domain}(P'), \mathrm{OrderFunction}(P)(x, y) = \mathrm{OrderFunction}(P')(x, y)$,
- $\forall x, y \in \mathrm{DisjointCopy}(i), \mathrm{OrderFunction}(P)(x, y) = \mathrm{OrderFunction}(\mathrm{DisjointCopy}(i))(x, y)$,
- $\forall x \in \mathrm{DisjointCopy}(i), \forall y \in \mathrm{DisjointCopy}(j), \mathrm{OrderFunction}(P)(x, y) =$ '$\sim$',
- $\forall x \in \mathrm{DisjointCopy}(i), \forall y \in \mathrm{Domain}(P'), \mathrm{OrderFunction}(P)(x, y) =$ '$<$' if $i = f^{-1}(y)$ or $f(i) < y (\mathrm{OrderFunction}(P')(f(i), y) \in \{=, <\})$, '$\sim$' otherwise .

Clearly, each element $f(i)$ of $\mathrm{Domain}(P')$ is now on a distinct level, since $\mathrm{DisjointCopy}(i)$ is a longest chain below it. Moreover, if some element in $\mathrm{DisjointCopy}(i)$ is less than two elements on the same level, then clearly, $f(i)$ must be less than these two elements, and this is impossible since $f(i)$ may only be less than elements in $\mathrm{Domain}(P')$, that are now scattered.

■

Theorem 3.5. *An ali order is either*

- *a well-founded total order,*
- *an antichain of size 2,*
- *the disjoint union of a well-founded chain and an incomparable element, where the well-founded chain has height 2 or is isomorphic to a regular cardinal/ordinal or its successor,*
- *or the order composition of two incomparable elements and a well-founded chain (we call this case a "2-based chain").*

Proof:

It is trivial to see that a well-founded total order is an ali order.

A well-founded chain and an incomparable element may or may not form an ali order.

- Clearly an antichain of size 2 is an ali order.
- A chain of height 2 and an incomparable element is an ali order: Indeed, consider x, y, z with $x < y, x \sim z, y \sim z$.
  - If $\mathrm{Level}_P(x) = \mathrm{Level}_P(z)$, there is nothing to do.
  - If $\mathrm{Level}_P(x) < \mathrm{Level}_P(z)$, then there is another element t such that $\mathrm{Level}_P(x) = \mathrm{Level}_P(t)$, and $t < z$ (since there is an element in each level below z, such that z is more than it, and $x \sim z$). Clearly, t, z, x give a level-induced suborder isomorphic to a chain of height 2 and an incomparable element.
  - If $\mathrm{Level}_P(x) > \mathrm{Level}_P(z)$, then there is another element t such that $\mathrm{Level}_P(z) = \mathrm{Level}_P(t)$, and $t < x$ (since there is an element in each level below x, such that x is more than it). Clearly, t, x, z give a level-induced suborder isomorphic to a chain of height 2 and an incomparable element. *Note that the same argument applies to well-founded chains of any height with a lowest element x and an incomparable element z.*
- A well-founded chain corresponding to a successor of a successor ordinal $\alpha+2$ more than 2 ($\alpha > 0$) and an incomparable element is not an ali order. (Such a chain is ended by a chain of height 2 on two consecutive levels α and $\alpha + 1$.) Indeed, consider the order P with $\mathrm{Domain}(P) = \mathrm{DisjointCopy}(\alpha + 2) \sqcup \mathrm{DisjointCopy}(\alpha + 1)$, such that:
  - $\forall x, y \in \mathrm{DisjointCopy}(\alpha{+}2), \mathrm{OrderFunction}(P)(x, y) = \mathrm{OrderFunction}(\mathrm{DisjointCopy}(\alpha+2))(x, y)$,
  - $\forall x, y \in \mathrm{DisjointCopy}(\alpha{+}1), \mathrm{OrderFunction}(P)(x, y) = \mathrm{OrderFunction}(\mathrm{DisjointCopy}(\alpha+1))(x, y)$,
  - $\forall x \in \mathrm{DisjointCopy}(\alpha{+}1), \forall y \in \mathrm{DisjointCopy}(\alpha{+}2), \mathrm{OrderFunction}(P)(x, y) =$ ‘ $<$ ’ if x corresponds to an ordinal less than y and x does not correspond to α, ‘ $\sim$ ’ otherwise .

  Any chain in P isomorphic to $\alpha + 2$ must have its greatest element to be the greatest element corresponding to $\alpha+1$ in $\mathrm{DisjointCopy}(\alpha+2)$. Then clearly the only element incomparable with it is the greatest element corresponding to α in $\mathrm{DisjointCopy}(\alpha + 1)$. Thus, since $\alpha > 0$ it cannot yield a level-induced suborder.
- A well-founded chain corresponding to (a successor of) a limit ordinal α that is singular (not a regular cardinal/ordinal) and an incomparable element is not an ali order. (Such a chain is isomorphic to $\gamma = \alpha$ (resp. $\gamma = \alpha + 1$).) Since α is not a regular cardinal, let $\beta + 1 < \gamma$ be a successor ordinal such that $\alpha \setminus \beta$ is not isomorphic to α. Now, consider the order P with $\mathrm{Domain}(P) = \mathrm{DisjointCopy}(\gamma) \sqcup \mathrm{DisjointCopy}(\beta + 1)$, such that:
  - $\forall x, y \in \mathrm{DisjointCopy}(\gamma), \mathrm{OrderFunction}(P)(x, y) = \mathrm{OrderFunction}(\mathrm{DisjointCopy}(\gamma))(x, y)$,
  - $\forall x, y \in \mathrm{DisjointCopy}(\beta{+}1), \mathrm{OrderFunction}(P)(x, y) = \mathrm{OrderFunction}(\mathrm{DisjointCopy}(\beta+1))(x, y)$,
  - $\forall x \in \mathrm{DisjointCopy}(\beta{+}1), \forall y \in \mathrm{DisjointCopy}(\gamma), \mathrm{OrderFunction}(P)(x, y) =$ ‘ $<$ ’ if x corresponds to an ordinal less than y and x does not correspond to β, ‘ $\sim$ ’ otherwise .

Since $\beta + 1 < \gamma$, any chain in P isomorphic to γ must have its final segment in $\mathrm{DisjointCopy}(\gamma)$. Then clearly the only element incomparable with it is the greatest element corresponding to β in $\mathrm{DisjointCopy}(\beta + 1)$. Thus, since $\beta > 0$, and $\alpha \setminus \beta$ is not isomorphic to α, it cannot yield a level-induced suborder.

- A well-founded chain corresponding to (a successor of) a regular limit ordinal/cardinal α and an incomparable element is an ali order. (Such a chain is isomorphic to $\gamma = \alpha$ (resp. $\gamma = \alpha + 1$).) Let $(x_i)_{i \in \gamma}$ and an incomparable element z form such an induced suborder in some partial order P. From what we noted for the case of a chain of height 2 and an incomparable element, we just need to consider the case where $\mathrm{Level}_P(x_0) < \mathrm{Level}_P(z)$. Clearly, there is a chain $(y_j)_{j \in \mathrm{Level}_P(z)}$ below z intersecting all levels below $\mathrm{Level}_P(z)$, such that, by transitivity, $x_i \not< y_j, i \in \gamma, j \in \mathrm{Level}_P(z)$.
  - If $\mathrm{Level}_P(z) \geq \mathrm{Supremum}(\mathrm{Level}_P(x_i), i \in \gamma)$, then $(y_j)_{j \in \mathrm{Level}_P(z), j \geq \mathrm{Level}_P(x_0)}$ contains a subchain isomorphic to γ, and $x_0 \sim y_j, i \in \gamma, j \in \mathrm{Level}_P(z), j \geq \mathrm{Level}_P(x_0)$. Taking the subchain starting on the same level than x_0 (made of the elements $(y_j)_{j \in \mathrm{Level}_P(z), j \geq \mathrm{Level}_P(x_0), j \in \{\mathrm{Level}(x_i), i \in \gamma\}}$), together with z on top of this subchain (if needed) and x_0, we obtain the sought level-induced suborder.
  - If $\mathrm{Level}_P(x_0) < \mathrm{Level}_P(z) < \mathrm{Supremum}(\mathrm{Level}_P(x_i), i \in \gamma)$, it is more complicated. If $(x_i)_{i \in \gamma, \mathrm{Level}_P(x_i) \geq \mathrm{Level}_P(z)}$ is reduced to the singleton x_α (hence $\gamma = \alpha + 1$), then we are in a situation equivalent to the previous case $\mathrm{Level}_P(z) \geq \mathrm{Supremum}(\mathrm{Level}_P(x_i), i \in \gamma)$, but we should replace it by $\mathrm{Level}_P(z) > \mathrm{Level}_P(x_i), \forall i \in \alpha$; it is clear in that case that $(y_j)_{j \in \mathrm{Level}_P(z), j \geq \mathrm{Level}_P(x_0)}$ contains a subchain isomorphic to α starting on the same level than x_0; together with z on top of this subchain and x_0, it yields the sought level-induced suborder.

    Assume that no element $x_i, i \in \gamma$ belongs to $\mathrm{Level}_P(z)$. There may be no element in $\mathrm{Level}_P(z)$ ordered with all elements $x_i, i \in \gamma$; nevertheless, there is a lowest element x_k such that $\mathrm{Level}_P(x_k) > \mathrm{Level}_P(z)$; and there is an element $x'_k < x_k$ such that $\mathrm{Level}_P(x'_k) = \mathrm{Level}_P(z)$. If there is an element $x_i, i \in \gamma$ that belongs to $\mathrm{Level}_P(z)$, we also name this element x'_k.

    Clearly, $\{x'_k\} \cup (x_i)_{i \in \gamma, \mathrm{Level}_P(x_i) \geq \mathrm{Level}_P(z)}$ is cofinal in a chain isomorphic to γ, and all elements of the chain are incomparable with z. Moreover, even in the case $\gamma = \alpha + 1$, we can remove the element x_α of this chain, and still obtain a chain cofinal in a chain isomorphic to α, since we already studied the case where $(x_i)_{i \in \gamma, \mathrm{Level}_P(x_i) \geq \mathrm{Level}_P(z)}$ is reduced to the singleton x_α. In both cases, by regularity of α, the subchain $\{x'_k\} \cup (x_i)_{i \in \gamma, \mathrm{Level}_P(x_i) \geq \mathrm{Level}_P(z)}$ is isomorphic to γ. Thus, together with the incomparable element z, it yields the sought level-induced suborder.

The only case left to study is then when the second element on level 0, y, is less than some element of the chain $(x_i)_{i \in \gamma}$. Let $(x_i)_{i \in \alpha}$ be the initial segment of elements that are incomparable with y.

- If $y < x_1, \alpha = 1$ and it is easy to see that such a well-founded chain with "triangular basis" or "2-based" is an ali order. Indeed, if $\text{Level}_P(x_0) < \text{Level}_P(y)$, there is an element in a chain with upper element y that intersects $\text{Level}_P(x_0)$; this element can be used instead of y in a level-induced suborder. The case $\text{Level}_P(x_0) > \text{Level}_P(y)$ is symetric.
- Let $\alpha > 1$. This time, this is not symetric.
  - If $\alpha > 1$ is a successor ordinal, then $(x_i)_{i\in\gamma} \sqcup \{y\}$ is not an ali order. We will lift y with a chain isomorphic to $\alpha - 1$. Consider the order P with $\text{Domain}(P) = (x_i)_{i\in\gamma} \sqcup \text{DisjointCopy}(\alpha - 1) \sqcup \{y\}$, such that:
    * $\forall x_j, x_k \in (x_i)_{i\in\gamma}, \text{OrderFunction}(P)(x_j, x_k) = \text{OrderFunction}(\gamma)(j, k)$,
    * $\forall u, v \in \text{DisjointCopy}(\alpha-1), \text{OrderFunction}(P)(u, v) = \text{OrderFunction}(\text{DisjointCopy}(\alpha-1))(u, v)$,
    * $\forall u \in \text{DisjointCopy}(\alpha-1), \forall v \in (x_i)_{i\in\gamma}, \text{OrderFunction}(P)(u, v) =$ '$<$' if u corresponds to an ordinal less than v, '$\sim$' otherwise ,
    * $\forall u \in \text{DisjointCopy}(\alpha - 1), \text{OrderFunction}(P)(u, y) =$ '$<$',
    * $\forall i \in \gamma\backslash\alpha, \text{OrderFunction}(P)(y, x_i) =$ '$<$', $\forall i \in \alpha, \text{OrderFunction}(P)(y, x_i) =$ '$\sim$'.

    Since no element of the chain $\text{DisjointCopy}(\alpha - 1) \sqcup \{y\}$ is more than two incomparable elements, and no element of the chain $(x_i)_{i\in\gamma}$ is less than an element in $\text{DisjointCopy}(\alpha-1)\sqcup\{y\}$, clearly, in a level-induced suborder, the chain isomorphic to $(x_i)_{i\in\gamma}$ must be a subchain of $(x_i)_{i\in\gamma}$, and the element that is incomparable with an initial segment of this subchain and less than the complementary final segment must be among $\text{DisjointCopy}(\alpha - 1) \sqcup \{y\}$. But it is now easy to check that for any choice among $\text{DisjointCopy}(\alpha - 1) \sqcup \{y\}$, and for any level above, the element of the main chain will be more than both elements in the level below; it will be isomorphic to a 2-based chain; it cannot yield a level-induced suborder isomorphic to $(x_i)_{i\in\gamma} \sqcup \{y\}$.
  - If α is a limit ordinal, then $(x_i)_{i\in\gamma} \sqcup \{y\}$ is not an ali order. We will lift y with a chain isomorphic to α; y will be the only element in level α, and $(x_i)_{i\in\gamma}$ "will be split" with a gap of one level between $(x_i)_{i\in\alpha}$ and $(x_i)_{i\in\gamma\backslash\alpha}$. Consider the order P with $\text{Domain}(P) = (x_i)_{i\in\gamma} \sqcup \text{DisjointCopy}(\alpha) \sqcup \{y\}$, such that:
    * $\forall x_j, x_k \in (x_i)_{i\in\gamma}, \text{OrderFunction}(P)(x_j, x_k) = \text{OrderFunction}(\gamma)(j, k)$,
    * $\forall u, v \in \text{DisjointCopy}(\alpha), \text{OrderFunction}(P)(u, v) = \text{OrderFunction}(\text{DisjointCopy}(\alpha))(u, v)$,
    * $\forall u \in \text{DisjointCopy}(\alpha), \forall v \in (x_i)_{i\in\gamma}, \text{OrderFunction}(P)(u, v) =$ '$<$' if u corresponds to an ordinal less than v, '$\sim$' otherwise ,
    * $\forall u \in \text{DisjointCopy}(\alpha), \text{OrderFunction}(P)(u, y) =$ '$<$',
    * $\forall i \in \gamma\backslash\alpha, \text{OrderFunction}(P)(y, x_i) =$ '$<$', $\forall i \in \alpha, \text{OrderFunction}(P)(y, x_i) =$ '$\sim$'.

    Since no element of the chain $\text{DisjointCopy}(\alpha) \sqcup \{y\}$ is more than two incomparable elements, and no element of the chain $(x_i)_{i\in\gamma}$ is less than an element in $\text{DisjointCopy}(\alpha) \sqcup \{y\}$, clearly, in a level-induced suborder, the chain isomorphic to $(x_i)_{i\in\gamma}$ must be a subchain of $(x_i)_{i\in\gamma}$, and the element that is incomparable with an initial segment of this subchain and less than the complementary final segment must be among $\mathrm{DisjointCopy}(\alpha) \sqcup \{y\}$. But it is now easy to check that for any choice among $\mathrm{DisjointCopy}(\alpha)$, and for any level above, the element of the main chain will be more than both elements in the level below; it will be isomorphic to a 2-based chain; it cannot yield a level-induced suborder isomorphic to $(x_i)_{i\in\gamma} \sqcup \{y\}$. There is still the choice y, but since y is now the only element in level α, it cannot yield a level-induced suborder.

This ends this technical proof. ■

Corollary 3.6. *An ali order is a* well-founded *order without induced suborder isomorphic to* $O_{obs1} = (\{a,b,c,d\}, \{a < b, c < d\}) \equiv \mathrm{Inv}(O_{obs1})$, $O_{obs2} = (\{a,b,c\}, \{a < b, a < c\})$, *an antichain of size 3, or* $(\{a,b,c,d\}, \{a < b, a < c, b < c, d < c\})$ *but infinite ali orders cannot be characterized by forbidden induced suborders. Thus, ali orders are a subclass of series parallel interval orders.*

Nevertheless, finite ali orders are the finite orders without induced suborder isomorphic to $O_{obs1} = (\{a,b,c,d\}, \{a < b, c < d\}) \equiv \mathrm{Inv}(O_{obs1})$, $O_{obs2} = (\{a,b,c\}, \{a < b, a < c\})$, *an antichain of size 3,* $(\{a,b,c,d\}, \{a < b, a < c, b < c\})$ *(a chain of height 3 and an incomparable element), or* $(\{a,b,c,d\}, \{a < b, a < c, b < c, d < c\})$. *Thus, finite ali orders are a subclass of the following classes: series parallel unit interval orders, semi-orders = unit interval orders, 1-weak orders (see Trenk (1998)).*

Thus finite ali orders can be recognized in time $O(n+m)$, where n is the number of elements and m is the number of comparability relationships, see the articles by Valdes et al. (1979) and Crespelle and Paul (2006), for example.

(Given a modular decomposition using disjoint sum and order composition, a simple tree-automaton can determine if it corresponds to an ali order and compute the length of the longest chain. There are two variants of modular decomposition:

- the binary one where binary disjoint sum and binary order composition have exactly two subtrees/subterms,
- the grouped one where grouped disjoint sum and grouped order composition may have more than two subtrees below, no two grouped disjoint sum nodes are adjacent in the decomposition, and no two grouped order composition nodes are adjacent in the decomposition.

The grouped variant can simplify some computations. Below, we precise when the computation applies only to one variant. Without loss of generality, we assume that on each node, we have a boolean bLeaf: true if the node is a leaf, false it is a disjoint sum node or an order composition node. The set of states of the tree-automaton has size 8 or 16, it is the cartesian product of 3 or 4 sets of substates:

- a boolean value bChain which is true if and only if the order defined by the modular decomposition up to this node is a chain/total order. bChain of a leaf/single element is true, bChain of a disjoint sum is false, bChain of an order composition is a logical conjunction (an AND) of the values of bChain for the subtrees of the order composition.

- a boolean value bAliOfHeight1 which is true if and only if the order defined by the modular decomposition up to this node is one or two incomparable elements. bAliOfHeight1 of a leaf/single element is true, bAliOfHeight1 of a disjoint sum is true if and only if there are exactly two suborders in the disjoint sum and both are leaves, bAliOfHeight1 of an order composition is false.

- a boolean value bAli2BasedChain (used primarily for binary modular decomposition) which is true if and only if the order defined by the modular decomposition up to this node is a 2-based chain. bAli2BasedChain of a leaf/single element is false, bAli2BasedChain of a disjoint sum is false, bAli2BasedChain of an order composition is true if and only if the first subtree has (bAli2BasedChain = 1 or (bChain = 0 and bAliOfHeight1 = 1)) and all other subtrees have bChain = 1.

- a boolean value bAli which is true if and only if the order defined by the modular decomposition up to this node is an ali order. bAli of a leaf/single element is true, bAli of a disjoint sum is true if and only if there are exactly two suborders in the disjoint sum, one is a chain (bChain = 1) of height 1 or 2 (counting if the number of elements of the subtree exceeds 2 can be done in constant time) and the other is a leaf (bLeaf = 1), bAli of a grouped order composition is true if and only if the first subtree has bChain = 1 or bAliOfHeight1 = 1 and all other subtrees have bChain = 1, bAli of a binary order composition is true if and only if the first subtree has bChain = 1 or bAliOfHeight1 = 1 or bAli2BasedChain = 1 and the second subtree has bChain = 1.

It may also be nice to compute:

- an integer value iLongestChain which is the number of elements of a longest chain in the order defined by the modular decomposition up to this node. iLongestChain of a leaf/single element is 1, iLongestChain of a disjoint sum is the maximum of the values of iLongestChain for the subtrees of the order composition, iLongestChain of an order composition is the sum of the values of iLongestChain for the subtrees of the order composition.

- a boolean value bAli2EndedChain (used primarily for binary modular decomposition) which is true if and only if the order defined by the modular decomposition up to this node is a 2-ended chain. bAli2EndedChain of a leaf/single element is false, bAli2EndedChain of a disjoint sum is false, bAli2EndedChain of an order composition is true if and only if the last subtree has (bAli2EndedChain = 1 or (bChain = 0 and bAliOfHeight1 = 1)) and all other subtrees have bChain = 1.

- a boolean value bAliInverse which is true if and only if the order defined by the modular decomposition up to this node is the inverse/reverse of an ali order. bAliInverse of a leaf/single element is true, bAliInverse of a disjoint sum is true if and only if there is exactly two suborders in the disjoint sum, one is a chain (bChain = 1) of height 1 or 2 (counting if the number of elements of the subtree exceeds 2 can be done in constant time) and the other is a leaf (bLeaf = 1), bAliInverse of a grouped order composition is true if and only if the last subtree has bChain = 1 or bAliOfHeight1 = 1 and all other subtrees have bChain = 1, bAliInverse of a binary order composition is true if and only if the second subtree has bChain = 1 or bAliOfHeight1 = 1 or bAli2EndedChain = 1 and the first subtree has bChain = 1.

Computing bChain, bAliOfHeight1, bAli2BasedChain/bAli2EndedChain, and bAli/bAliInverse on all nodes takes $O(n)$ time, computing iLongestChain on all nodes takes $O(n \times \log(n))$ time (or $O?(n)$ time on unit cost RAM-model, if there is less than 2^{64} elements which should be the case for efficient computations, the cost of maximum and sum computation is done by a constant number of hardware instructions on current hardware architectures, and there will be no empirical asymptotic difference, up to a constant factor, between a $O(n)$ and $O?(n)$ algorithms with similar input/output profile (memory access matters a lot)).)

4 Orders that are naturally consecutive level-induced suborders

In this section, we assume that a given well-founded order P' is a level-induced suborder of a well-founded order P. We study necessary and sufficient conditions on P' to have that P' is a consecutive level-induced suborder of P.

Definition 4.1 (nacli orders)**.** *A* nacli order *P' is a well-founded order such that whenever P' is isomorphic to a level-induced suborder of a well-founded order P, then P' is also isomorphic to a consecutive level-induced suborder of P.*

We first observe that :

Lemma 4.2. *For any well-founded order P' containing an induced suborder isomorphic to $O_{obs2} = (\{a, b, c\}, \{a < b, a < c\})$ or $O_{obs3} = (\{a, b, c\}, \{a > b, a > c\})$, there is a well-founded order P such that P' is a level-induced suborder of P, but P' is not isomorphic to any consecutive level-induced suborder of P.*

Proof:

We use a disjoint sum of well-founded chains of same height to lift each level of P' so that any two levels of P' are now γ levels apart, where $\gamma \geq \omega_{\beta+1}$, and the cardinal of $\mathrm{Domain}(P')$ is at most $\aleph_\beta$. Again by Zermelo's axiom, there is a bijection f between some ordinal α and $\mathrm{Domain}(P')$. We add a distinct well-founded chain for each element of $\mathrm{Domain}(P')$. Let $\mathrm{DisjointCopy}(i)$ be a chain isomorphic to the ordinal i such that its elements are assumed to be distinct from all other elements considered in the following formula: $\mathrm{Domain}(P) = \mathrm{Domain}(P') \sqcup (\bigsqcup_{i \in \alpha} \mathrm{DisjointCopy}(\gamma \times \mathrm{Level}_{P'}(f(i))))$.

- $\forall x, y \in \mathrm{Domain}(P'), \mathrm{OrderFunction}(P)(x, y) = \mathrm{OrderFunction}(P')(x, y)$,
- $\forall x, y \in \mathrm{DisjointCopy}(\gamma \times \mathrm{Level}_{P'}(f(i))), \mathrm{OrderFunction}(P)(x, y) = \mathrm{OrderFunction}(\mathrm{DisjointCopy}(\gamma \times \mathrm{Level}_{P'}(f(i))))(x, y)$,

- $\forall x \in \mathrm{DisjointCopy}(\gamma \times Level_{P'}(f(i))), \forall y \in \mathrm{DisjointCopy}(\gamma \times \mathrm{Level}_{P'}(f(j))), \mathrm{OrderFunction}(P)(x,y) =$ '$\sim$',
- $\forall x \in \mathrm{DisjointCopy}(\gamma \times \mathrm{Level}_{P'}(f(i))), \forall y \in \mathrm{Domain}(P'), \mathrm{OrderFunction}(P)(x,y) =$ '$<$' if $i = f^{-1}(y)$ or $f(i) < y(\mathrm{OrderFunction}(P')(f(i),y) \in \{=,<\})$, '$\sim$' otherwise .

Clearly, P' is a level-induced suborder of P, and any two levels of P' are now γ levels apart, since $\mathrm{DisjointCopy}(\gamma \times \mathrm{Level}_{P'}(f(i)))$ is a longest chain below element $f(i)$.

Let (x,y,z) be a triple of elements of $\mathrm{Domain}(P)$, such that $x < y, x < z, y \sim z$ (O_{obs2}), or $x > y, x > z, y \sim z$ (O_{obs3}). It naturally defines one ordinal $\mathrm{gap}(x,y,z)$: the supremum of the ordinals corresponding to a chain between x and y, or between x and z.

Observe that no element of $\mathrm{DisjointCopy}(\gamma \times \mathrm{Level}_{P'}(f(i)))$ is more than an element, unless that element is also in $\mathrm{DisjointCopy}(\gamma \times \mathrm{Level}_{P'}(f(i)))$. Hence, it cannot be more than two incomparable elements.

Clearly, if it is less than two incomparable elements like x, then these two elements are in $\mathrm{Domain}(P')$, and $x \in \{f(i)\} \sqcup \mathrm{DisjointCopy}(\gamma \times \mathrm{Level}_{P'}(f(i)))$ implies that $f(i)$ is also less than these two elements. Moreover, $\mathrm{gap}(x,y,z) \geq \mathrm{gap}(f(i),y,z) \geq \gamma$, in that case.

Thus, if there is an induced suborder isomorphic to $O_{obs2} = (\{a,b,c\},\{a < b, a < c\})$ in P', then no consecutive level-induced suborder isomorphic to P' exists in P, because of the ordinal gap in P between original levels of P' that is superior to the ordinal corresponding to the cardinal of P'.

Otherwise, there is an induced suborder isomorphic to $O_{obs3} = (\{a,b,c\},\{a > b, a > c\})$ in P'. Consider such an induced suborder (x,y,z) in P. We already noted that x must be in P'; if both y,z are in $\mathrm{DisjointCopy}(\gamma \times \mathrm{Level}_{P'}(f(k)))$, then they are comparable, a contradiction. Hence, without loss of generality, $y \in \{f(j)\} \sqcup \mathrm{DisjointCopy}(\gamma \times \mathrm{Level}_{P'}(f(j)))$, for some $f(j) \neq x, f(j) \in \mathrm{Domain}(P')$. It is now trivial to see that $(\{x,f(j),z\},\{x > f(j), x > z\})$ is also an induced suborder isomorphic to $O_{obs3} = (\{a,b,c\},\{a > b, a > c\})$ with $\mathrm{gap}(x,y,z) \geq \mathrm{gap}(x,f(j),z)$. But since $\mathrm{gap}(x,f(j),z) \geq \gamma$, again we have that no consecutive level-induced suborder isomorphic to P' exists in P.

■

Corollary 4.3. *A nacli order is the disjoint union of well-founded chains.*

Lemma 4.4. *No nacli order has more than one level of size at least 2.*

Proof:

Again, we create a gap between consecutive levels of P'. We use a unique well-founded chain of height $\gamma \times \mathrm{Height}(P')$ to lift all levels of P' so that any two levels of P' are now γ levels apart, where $\gamma \geq \omega_{\beta+1}$, and the cardinal of $\mathrm{Domain}(P')$ is at most $\aleph_\beta$. $\mathrm{Domain}(P) = \mathrm{Domain}(P') \sqcup \mathrm{DisjointCopy}(\gamma \times \mathrm{Height}(P'))$. Since added levels have size 1 and original levels are too far apart, at most one level can have size more than one in a consecutive level-induced suborder.

■

Theorem 4.5. *A nacli order is a well-founded chain, an antichain, or the disjoint union of a well-founded chain and an antichain. Equivalently, a nacli order is a* well-founded *order without induced suborder isomorphic to* $O_{obs1} = (\{a, b, c, d\}, \{a < b, c < d\}) \equiv \mathrm{Inv}(O_{obs1})$, $O_{obs2} = (\{a, b, c\}, \{a < b, a < c\})$, *or* $O_{obs3} = (\{a, b, c\}, \{a > b, a > c\})$. *In particular, nacli orders are a subclass of series parallel interval orders, and all ali orders except 2-based chains are also nacli orders.*

Proof:

By previous lemmas, only well-founded chains, antichains, or the disjoint unions of a well-founded chain and an antichain may be nacli orders. The proof by transfinite induction that such orders are indeed nacli orders is trivial. In any superorder, fix the first level of the disjoint union of a well-founded chain and an antichain and close the gap with the second level, then close the gap between the second and third level, etc. Everything follows from transitivity and the fact that a single well-founded chain can not be lifted by another suborder that does not contain an isomorphic chain.

■

Thus finite nacli orders can be recognized in time $O(n+m)$ with techniques similar to the end of previous section. (bNacliOfHeight1 = bAntichain is the logical conjunction of bNacliOfHeight1 of subtrees on disjoint sums nodes, and false on order compositions nodes. bNacli is the logical conjunction of bChain on order compositions nodes, and it is true on disjoint sums nodes if and only if bNacliOfHeight1 is true on all subtrees, except maybe at most one where instead bChain is true (grouped case), or (bChain or bNacli) is true (binary case).)

5 Algorithms to find ali induced suborders and nacli level-induced suborders

All orders in this section are finite, hence well-founded. We first start with the simple case of chains and orders made of a chain of height at most 2 and an incomparable element, i.e. orders that are ali orders and nacli orders at the same time. Assume we want to find such an order in a superorder P, where n is the cardinal of $\mathrm{Domain}(P)$, and m is the number of comparability relationships in P. We first compute a level decomposition of P in time $O(n^3)$, with the additional constraint that we store in each element a reference to another element that is less than it in the *previous* level. We do it as follow, once an element has been selected to be added in the current level, for all elements that are greater than it overwrite their reference with the selected element. Clearly, the last overwrite will be in the previous level. It is easy to see that this can be done in time $O(m) \leq O(n^2)$. Let h be the height of P, and s be the size of the longest chain in the ali and nacli suborder.

- If the suborder is a chain,
  - if $h \geq s$, take any element x in the level $s-1$,
  - otherwise there is no such ((consecutive) level-)induced suborder.
- Otherwise, for any element x in the level l ranging from $s-1$ to $h-1$ (first loop), check if there is an element y in level $l-s+1$ (second loop) that is incomparable with x. If no check succeeds, there is no such ((consecutive) level-)induced suborder (this check is sufficient by transitivity).

If you got an x and optionnaly a corresponding y, then you can output the consecutive level-induced suborder made of element y and the chain obtained by following the references set during the level decomposition, starting from element x and iterating $s-1$ times. Clearly, these two loops take time $O(n^2)$. Thus, whatever the size of an ali and nacli order, finding such a ((consecutive) level-)induced suborder has time complexity in $O(n^3)$.

We now look at the odd case of the 2-based chains that are ali orders but not nacli orders. Assume we want to find such a 2-based chain in a superorder P. We first compute a level decomposition of P in time $O(n^3)$. Then, in time $O(m \times \log(n))$, proceeding from the last level to the first level, we can compute on each element x the size $slc(x)$ of the longest chain starting with x: this size is 1 if no element is greater and the maximum plus one of $slc(y)$ for y greater than x otherwise, for backtracking purpose, we keep a reference to such an y that gave the maximum for each x. Then, for each element x such that $slc(x)$ is at least the size of the 2-based chain minus two, we can enumerate all pairs of elements in a same level below such that both elements are less than x; it takes time $O(n^3)$. Thus, whatever the size of a 2-based chain, finding such an induced suborder has time complexity in $O(n^3)$. The same result applies to 2-ended chains by considering the inverse order.

In order to find (consecutive) level-induced suborders that are nacli orders, we just modify the algorithm for ali and nacli suborders as follow: Let r be the number of elements of the first level of the suborder minus one. Replace

- "Otherwise, for any element x in the level l ranging from $s-1$ to $h-1$ (first loop), check if there is an element y in level $l-s+1$ (second loop) that is incomparable with x. If no check succeeds, there is no such ((consecutive) level-)induced suborder (this check is sufficient by transitivity)."

by

- "Otherwise, for any element x in the level l ranging from $s-1$ to $h-1$ (first loop), check if there are r elements $y_1, \ldots, y_r$ in level $l-s+1$ (second loop) that are incomparable with x. If no check succeeds, there is no such (consecutive) level-induced suborder (this check is sufficient by transitivity)."

6 Conclusion

Maybe we should talk about partial-level-induced suborders, since we do not impose to keep all elements of a level of a superorder. However, in that case total/global-level-induced suborders would be rather restricted. And no similar results could be obtained unless considering superorders of bounded level-width.

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