Partitions versus sets : a case of duality

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Abstract

In a recent paper, Amini et al. introduce a general framework to prove duality theorems between special decompositions and their dual combinatorial object. They thus unify all known ad-hoc proofs in one single theorem. While this unification process is definitely good, their main theorem remains quite technical and does not give a real insight of why some decompositions admit dual objects and why others do not. The goal of this paper is both to generalise a little this framework and to give an enlightening simple proof of its central theorem.

1 Introduction

In the last 30 years, many decompositions on graphs and more general structures such as tree-decompositions, branch-decompositions of graphs [RS84,RS91]

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and matroids [HW06,OS07] have been defined. Most of those decompositions were later found dual combinatorial objects (brambles, tangles...). Those objects are duals in that a decomposition exists if and only if the dual object does not.

In [AMNT08], the authors present a general framework to prove those kind of dual relations. The framework is the following. A *partitioning tree* on a finite set E is a tree T whose leaves are identified with the elements of E in a one-to-one way. Every internal node v of T corresponds to the partition of E whose parts are the leaves of the subtrees obtained by deleting v. A partitioning tree T is *compatible* with a set of partitions $\mathcal{P}$ of E if all the *node partitions* of T belong to $\mathcal{P}$. By carefully choosing the set $\mathcal{P}$, one gets classical decompositions. For example, let $G = (V, E)$ be a graph. The *border* of a partition μ of E is the set of vertices incident with edges in at least two parts of μ. For every integer k, let $\mathcal{P}_k$ be the set of partitions whose borders contain at most $k+1$ vertices. A partitioning tree is compatible with $\mathcal{P}_k$ if and only if $\text{tw}(G) \leq k$. Tree-width thus corresponds to a class of sets of partitions.

The dual objects of partitioning trees are *brambles*. A *non-principal $\mathcal{P}$-bramble* is a nonempty set of pairwise intersecting subsets of E that contains no singleton and which contains a part of every partition in $\mathcal{P}$. Non-principal $\mathcal{P}$-brambles and partitioning trees cannot coexist but there may be none of them. In this framework, the duality theorem between tree-decompositions and brambles becomes: for any graph G and any k, there exists a partitioning-tree compatible with $\mathcal{P}_k$ if and only if no non-principal $\mathcal{P}_k$-bramble exists.

The authors try to characterise classes $\mathcal{C}$ of sets of partitions such that for any $\mathcal{P} \in \mathcal{C}$, there is a partitioning tree compatible with $\mathcal{P}$ if and only if no $\mathcal{P}$-bramble does. To do so, they define classes of partition by the mean of weight functions on partitions, and they prove that if a function is (*weakly*) *submodular*, then the corresponding class has the required property. For example, the weight function corresponding to tree-width (the size of the border of a partition) is submodular. This way the authors prove all known duality theorem to date.

While [AMNT08]'s framework unifies many ad-hoc proofs of duality between decompositions and their dual objects, its core theorem mimics a proof of [RS91]. This proof is quite technical and does not give a real insight of why a class of partitions leads to duality between partitioning trees and $\mathcal{P}$-brambles. Moreover, at least one partition function, the function $\max_f$ that corresponds to branch-width, is not weakly-submodular. Since this function is a limit of weakly-submodular functions, Amini et al. manage to also apply their theorem to branch-width but this is not really satisfying.

The goal of this paper is twofold. First we give an easy proof of the duality

theorem, then we slightly extend the definition of *weak submodularity* so that the function $\max_f$ becomes weakly-submodular.

To do so, we consider *partial partitioning trees* by labelling the leaves of a tree with the elements of any partition of E (the partitioning trees are the partial partitioning trees whose *displayed partition* is made of all singletons). The set $\mathcal{P}^\uparrow$ then denote the set of all displayed partitions that arise from partial partitioning trees compatible with $\mathcal{P}$. We do not make any distinction between principal and non principal $\mathcal{P}$-brambles. Instead we define a *set of small sets* to be a subset of 2^E closed under taking subset, and whose elements are *small*. We say that a set of partitions $\mathcal{P}^\uparrow$ is *dualising* if for any set of small sets $\mathcal{S}$, there exists a *big bramble* (i.e. a bramble containing no part in $\mathcal{S}$) if and only if $\mathcal{P}^\uparrow$ contains no *small partition* (i.e a partition whose parts all belong to $\mathcal{S}$). Thus the classical results of duality are the sub-case where $\mathcal{S}$ is made of the empty set and the singletons. Note that since a $\mathcal{P}$-bramble Br meets all partitions in $\mathcal{P}$, if $\mathcal{P}$ contains a small partition, Br cannot contain only big parts ; a class of partitions cannot admit a big bramble and contain a small partition.

In Section 2, we fix some notations and give basic definitions. In Section 3, we give an equivalent and yet easier notion to duality: refinement. In Section 4, we give a sufficient condition on $\mathcal{P}$ so that $\mathcal{P}^\uparrow$ is refining (and thus dualising). Finally, in Section 5, we extend the definition of *weak-submodularity* to match our sufficient condition for duality, and we prove that the partition function $\max_f$ is weakly-submodular and thus, that branch-width fully belongs to the unifying framework.

2 Preliminaries

In this paper, E is a fixed set with at least two elements, 2^E is the set of subsets of E, and $\mathcal{P}$ is a set of partitions of E. Greek letters α, β, ... denote sets of subsets of E while capital letters A, B, ... denote subsets of E. The capital letters I, J are special in that they denote sets of indices. We write X^c for the complement $E \setminus X$ of X. We denote a finite union $\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_p$ by $(\alpha_1|\alpha_2|\ldots|\alpha_p)$ and also shorten $(\{A\}|\alpha|\{B\})$ in $(A|\alpha|B)$.

Let $\alpha = \{A_i \; ; \; i \in I\}$ and $\beta = \{B_j \; ; \; j \in J\}$ be subsets of 2^E. We say that α is *smaller* than β if $\cup\alpha = \cup\beta$ and $|I| \leq |J|$, and α is *finer* than β if $\cup\alpha = \cup\beta$ and there exists a one to one[3] mapping $f : I \mapsto J$ such that $A_i \subseteq B_{f(i)}$, for all $i \in I$. Note that if α is finer than β it is also smaller than β. When we say that $(\alpha_1|\ldots|\alpha_p)$ is finer or equal to $(\beta_1|\ldots|\beta_q)$, we mean that $p \leq q$ and each

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[3] The one to one requirement is not mandatory but it allows easier proofs.

α_i is finer or equal to β_i. For any F, $\alpha \setminus F$ denotes the set $\{A_i \setminus F; i \in I\}$. If α is a covering of E, the *overlap* of α is the set $\mathrm{ov}(\alpha)$ of the elements that belong to at least two parts of α. A covering is a partition if and only if its overlap is empty, and if α is finer than β and $\mathrm{ov}(\alpha) \subsetneq \mathrm{ov}(\beta)$, then it is easy to see that α is strictly finer than β.

Note that the set $\mathcal{P}^\uparrow$ of partitions labelling partial partitioning trees is exactly the smallest superset of $\mathcal{P}$ such that if $(\alpha|A)$, $(A^c|\beta) \in \mathcal{P}^\uparrow$, then $(\alpha|\beta) \in \mathcal{P}^\uparrow$. Since the partitions in $\mathcal{P}^\uparrow$ come from partial partitionning tree, it is easy to see that for any $(\alpha|A) \in \mathcal{P}^\uparrow$, there exists $(\gamma|C) \in \mathcal{P}$ such that either $(\alpha|A) = (\gamma|C)$ or $(\alpha|A) = (\gamma|\mu|A)$ for some $(C^c|\mu|A) \in \mathcal{P}^\uparrow$. Such a $(\gamma|C)$ *decomposes* $(\alpha|A)$.

A $\mathcal{P}$-*bramble* or just *bramble* when no confusion can occur is a set Br of subsets of E such that

- Br contains a part of every $\mu \in \mathcal{P}$ (Br *meets* every $\mu \in \mathcal{P}$);
- the elements of Br are pairwise intersecting.

If Br is a $\mathcal{P}$-bramble, we say that $\mathcal{P}$ *admit* the bramble Br.

Remark 1 *A set Br is a $\mathcal{P}$-bramble if and only if it is a $\mathcal{P}^\uparrow$-bramble. Indeed, since the elements of Br are pairwise intersecting, if Br meets both $(\alpha|A)$, $(A^c|\beta) \in \mathcal{P}^\uparrow$, Br cannot contain A and A^c so it meets $(\alpha|\beta)$, the forward implication follows. The backward implication follows from $\mathcal{P} \subseteq \mathcal{P}^\uparrow$.*

We can thus freely speak of a bramble for either a $\mathcal{P}$-bramble or a $\mathcal{P}^\uparrow$-bramble. Moreover this remark may justify the definition of a bramble for an algorithmic search for an obstruction could be restricted to $\mathcal{P}$.

3 Dualising and refining sets of partitions

We now introduce the *refining* properties, and we prove that a set of partitions is refining if and only if it is dualising. In the end, we use this equivalence on $\mathcal{P}^\uparrow$ but since the proof is not specific to $\mathcal{P}^\uparrow$, we state it for $\mathcal{P}$.

A class $\mathcal{P}$ is *refining* if for any $(\alpha|A)$, $(B|\beta) \in \mathcal{P}$ with A and B disjoint, $\mathcal{P}$ contains a partition finer than the covering $(\alpha|\beta)$.

Theorem 2 *If $\mathcal{P}$ is refining, then $\mathcal{P}$ is dualising.*

PROOF. Suppose that $\mathcal{P}$ is refining and contains no small partition for some set of small sets. There exists a set closed under taking superset that contains

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a big part of every partition in $\mathcal{P}$. We claim that any such Br taken inclusion-wise minimal is a big bramble. If not, take A, B inclusion-wise minimal disjoint sets in Br. Since $Br \setminus \{A\}$ and $Br \setminus \{B\}$ are upward close and Br is minimal, there exists $(\alpha|A)$, $(B|\beta) \in \mathcal{P}$ such that Br does not meet $(\alpha|\beta)$. But $\mathcal{P}$ is refining and contains a λ finer than $(\alpha|\beta)$. Since Br is closed under taking superset, $\lambda \cap Br$ is empty, a contradiction. □

Conversely,

Theorem 3 *If $\mathcal{P}$ is dualising, then $\mathcal{P}$ is refining.*

PROOF. Suppose that $\mathcal{P}$ is not refining. Let $(\alpha|A)$, $(B|\beta) \in \mathcal{P}$ with A and B disjoint and such that $\mathcal{P}$ contains no partition finer than $(\alpha|\beta)$. Consider the small set "up to" $(\alpha|\beta)$, i.e. the set of all subsets of parts of $(\alpha|\beta)$. We claim that $\mathcal{P}$ contains no small partition and that no big bramble exists. Indeed

- Since they are finer than $(\alpha|\beta)$, $\mathcal{P}$ contains no small partition.
- Since a bramble cannot contain both A and B, it must contain a small set to meet both $(\alpha|A)$ and $(B|\beta)$. □

We would like to emphasise that we will only use Theorem 2 for this is indeed all what is needed to obtain the duality theorems between tree-decompositions and brambles.

We remark that in the proof of Theorem 2, the finer order need not be defined with one to one mapping. Thus it is a corollary of Theorems 2 and 3 that the refining property is equivalent if it is defined using a finer order with arbitrary mapping.

4 Pushing sets of partitions

We now introduce a property on $\mathcal{P}$ that ensures that $\mathcal{P}^{\uparrow}$ is refining and thus, by Theorem 2, that $\mathcal{P}^{\uparrow}$ is dualising.

A set of partition $\mathcal{P}$ is *pushing* if for every pair of partitions $(\alpha|A)$ and $(B|\beta)$ in $\mathcal{P}$ with $A^c \cap B^c \neq \emptyset$, there exists a non empty $F \subseteq A^c \cap B^c$ with $(\alpha \setminus F|A \cup F) \in \mathcal{P}$ or $(B \cup F|\beta \setminus F) \in \mathcal{P}$.

Theorem 4 *If $\mathcal{P}$ is pushing, then $\mathcal{P}^{\uparrow}$ is refining.*

PROOF. Suppose for a contradiction that $\mathcal{P}$ is pushing, $(\alpha|A)$, $(B|\beta)$ belong to $\mathcal{P}^{\uparrow}$ with A and B disjoint, and yet $\mathcal{P}^{\uparrow}$ contain no partition finer than $(\alpha|\beta)$. Choose $(\alpha|\beta)$ smallest and then finest. Let $O = \text{ov}((\alpha|\beta))$.

Let $(\gamma|C)$ and $(D|\delta)$ decompose $(\alpha|A)$ and $(B|\beta)$. Clearly $C^c \cap D^c \subseteq O$. We claim that $O \subseteq C^c \cap D^c$. Indeed suppose that, say, $O \not\subseteq C^c$. Since $O \subseteq A^c$, clearly, $(\gamma|C) \neq (\alpha|A)$. Let $(C^c|\mu|A) \in \mathcal{P}^{\uparrow}$ be such that $(\gamma|\mu|A) = (\alpha|A)$. Since $(C^c|\mu|A)$ is smaller than $(\alpha|A)$, there exists $(C'|\mu'|\beta') \in \mathcal{P}^{\uparrow}$ finer than $(C^c|\mu|\beta)$. Since $\text{ov}((\gamma|\mu'|\beta')) \subseteq C^c$, $(\gamma|\mu'|\beta')$ is strictly finer than $(\alpha|\beta)$, a contradiction.

Consider $(\gamma|C)$ and $(D|\delta)$. Since $\mathcal{P}$ is pushing and $C^c \cap D^c = O$ is non empty, let $F \subseteq O$ be non empty such that say, $(\gamma \setminus F, C \cup F) \in \mathcal{P}$.

- If $(\gamma|C) = (\alpha|A)$, then $(\gamma \setminus F|\beta)$ is strictly finer than $(\alpha|\beta)$, a contradiction.
- If $(\gamma|C) \neq (\alpha|A)$, let $(C^c|\mu|A) \in \mathcal{P}^{\uparrow}$ with $(\gamma|\mu|A) = (\alpha|A)$. Since $(C^c|\mu|A)$ is smaller than $(\alpha|A)$, there exists $(C'|\mu'|\beta') \in \mathcal{P}^{\uparrow}$ finer than $(C^c|\mu|\beta)$. If $O \not\subseteq C'^c$, then $(\gamma|\mu'|\beta')$ is strictly finer than $(\alpha|\beta)$, a contradiction. The set $(\gamma \setminus F|\mu'|\beta')$ is thus a covering of E and since its overlap is a subset of $O \setminus F$, it is strictly finer than $(\alpha|\beta)$, a contradiction. □

5 Submodular partition functions

A *partition function* is a function from the set of partitions of E to $\mathbb{R} \cup \{+\infty\}$. In [AMNT08], the authors define *weak submodular* partition functions as partition functions such that for every partitions $(\alpha|A)$ and $(B|\beta)$, at least one of the following holds:

- there exists $A \subset F \subseteq (B \setminus A)^c$ with $\Psi((\alpha|A)) > \Psi((\alpha \setminus F|A \cup F))$;
- $\Psi((\beta|B)) \geq \Psi((\beta \setminus A^c|B \cup A^c))$.

Since $(\beta|B)$ and $(\beta \setminus A^c|B \cup A^c)$ are equal when $A^c \cap B^c = \emptyset$, this definition is only really interesting when $A^c \cap B^c \neq \emptyset$. We define *weak submodular* partition functions as partition functions such that for every $(\alpha|A)$ and $(B|\beta)$ with $A^c \cap B^c \neq \emptyset$, there exists a non empty $F \subseteq A^c \cap B^c$ such that at least one of the following holds:

- $\Psi((\alpha|A)) \geq \Psi((\alpha \setminus F|A \cup F))$;
- $\Psi((\beta|B)) \geq \Psi((\beta \setminus F|B \cup F))$.

This definition indeed generalises the previous one.

- Suppose that there exists $A \subset F \subseteq (B \setminus A)^c$ with $\Psi((\alpha|A)) > \Psi((\alpha \setminus F|A \cup F))$. Set $F' := F \cap (A^c \cap B^c)$. Since $F = F' \cup A$, $(\alpha \setminus F|A \cup F) = (\alpha \setminus F'|A \cup F')$.

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Thus $\Psi((\alpha|A)) > \Psi((\alpha \setminus F'|A \cup F'))$ with F' non empty.

- Suppose that $\Psi((\beta|B)) \geq \Psi((\beta \setminus A^c|B \cup A^c))$. Set $F := A^c \cap B^c$. Since $(\beta \setminus A^c|B \cup A^c) = (\beta \setminus F|B \cup F)$, $\Psi((\beta|B)) \geq \Psi((\beta \setminus F|B \cup F))$ and F is non empty.

It is obvious that given a weak submodular partition function Ψ the class of partitions $\mathcal{P} = \{\alpha ; \Psi(\alpha) \leq r\}$, for some $r \in \mathbb{R}$, is pushing. Conversely if $\mathcal{P}$ is pushing, then defining Ψ as $\Psi(\alpha) = 0$ if $\alpha \in \mathcal{P}$ and $\Psi(\alpha) = 1$ otherwise, we obtain a weak submodular partition function.

A *connectivity function* is a function $f : 2^E \mapsto \mathbb{R} \cup \{+\infty\}$ which is symmetric (i.e. for any $A \subseteq E$, $f(A) = f(A^c)$) and submodular (i.e. for any A, $B \subseteq E$, $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$). For any connectivity function f, we define the partition function $\max_f$ by $\max_f(\alpha) = \max\{f(A) ; A \in \alpha\}$ (α a partition of E).

Lemma 5 *The function $\max_f$ is a weakly submodular partition function.*

PROOF. Let $(\alpha|A)$ and $(B|\beta)$ be two partitions of E such that $A^c \cap B^c$ is non empty. Let F be such that $A \setminus B \subseteq F \subseteq (B \setminus A)^c$ such that $f(F)$ is minimum. We claim that $\max_f((\alpha|A)) \geq \max_f((\alpha \setminus F|A \cup F))$.

Indeed, since, $F' = F \cap A$ is allowed, $f(F \cap A) \geq f(F)$, and by submodularity, since $f(F) + f(A) \geq f(A \cap F) + f(A \cup F)$, we have $f(A) \geq f(A \cup F)$. For every $X \in \alpha$, we have by submodularity of f:

$$f(X) + f(F^c) \geq f(X \cap F^c) + f(X \cup F^c) \tag{1}$$

Since $f(F)$ is minimum, $f(F) \leq f(F \setminus X)$, and thus f being symmetric:

$$f(X \cup F^c) \geq f(F^c) \tag{2}$$

Adding (1) and (2), we obtain $f(X) \geq f(X \cap F^c)$. Thus $\max_f((\alpha|A)) \geq \max_f((\alpha \setminus F, A \cup F))$, as claimed.

Similarly, $\max_f((B|\beta)) \geq \max_f((B \cup F^c|\beta \setminus F^c))$. Now at least one of $F_A := F \cap (A^c \cap B^c)$ and $F_B := F^c \cap (A^c \cap B^c)$, say F_A, is non empty. Since $(\alpha \setminus F|A \cup F) = (\alpha \setminus F_A|A \cup F_A)$, there exists a non empty $F_A \subseteq A^c \cap B^c$ with $\max_f((\alpha|A)) \geq \max_f((\alpha \setminus F_A, A \cup F_A))$ which proves that $\max_f$ is weakly submodular. $\square$

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