

First difference principle applied to twin-width and merge-width of binary structures

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Abstract

In this article, we study twin-width and merge-width of binary structures under the light of first difference principle.

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1 Introduction

In this article, we continue our work of contextualisation of many graphs or binary structures widths in the framework of first difference principle. This work started in Lyaudet (2020). Before that, following Cantor (1895), Hausdorff (1907) et Sierpiński (1932), we started to study the first difference principle in Lyaudet (2018), Lyaudet (2019). We show that *twin-width* (introduced in Bonnet et al. (2020)) and *merge-width* (introduced in Dreier and Toruńczyk (2025)) fit very naturally in the framework of first difference principle.

2 Common definitions

For twin-width as well as for merge-width, we need the first difference principle extended with the special constant adjacency type *nyf* (*not yet fixed*). (There is also a special constant adjacency type *alf* (*already fixed*). We published both for the first time in Lyaudet (2020), but the idea of these two special adjacency types dates from the end of 2019, and *alf* is implicit in the definition of tree-questionable-width in Lyaudet (2019).)

Consider a binary signature \mathcal{S} of unary and binary relations and functions. Given a set S , an (\mathcal{S}, S, k, l) -*mapping-run* is an (ordinal-indexed) sequence $(S_i)_{i \in I}$ of length l of \mathcal{S} -structures of cardinality at most k , k being the lowest such cardinal, together with a sequence of mappings f_i from S to the domains of S_i structures. Each S_i is an

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\mathcal{S} -structure-item (of the mapping-run/the $(S_i)_{i \in l}$ sequence). Each element/vertex x of S is thus associated to a sequence $(x_i = f_i(x))_{i \in l}$; this sequence is an “element” of the mapping-run/the $(S_i)_{i \in l}$ sequence; it corresponds to an element/vertex of S ; each $x_i = f_i(x)$ is an *element/vertex-item*.

Definition 2.1 (nyf-extended question). *Given two elements X, Y of a mapping-run, we say that (q, x_q, y_q, S_q) is the question of X, Y , if q is the smallest ordinal such that $x_q \neq y_q$ and the adjacency type between x_q and y_q is not nyf.*

3 Twin-width

Definition 3.1 (Twin-width). *A twin-decomposition of an \mathcal{S} -structure S is a sequence of \mathcal{S} -structures indexed by an ordinal $(S_i)_{i \in l+1}$ such that $S_0 = S$ and S_l is a singleton. We always go from S_i to S_{i+1} by a unique merging/contraction of two vertices of S_i into one. Given two vertices $x, y \in S_i$, we say the edge between these vertices is red if the adjacencies in the starting binary structure are of at least two distinct types between the vertices corresponding to $f_i^{-1}(x)$ and those corresponding to $f_i^{-1}(y)$. The maximum red degree of an \mathcal{S} -structure S_i is equal to the supremum of the degrees of its vertices when we only keep the red edges. The width of a twin-decomposition $(S_i)_{i \in l+1}$ is equal to the supremum of the maximum red degrees of the S_i . The twin-width of an \mathcal{S} -structure S , denoted by $\text{tw}(S)$, is the minimum of the widths of all its twin-decompositions.*

If we do not focus on the size of the structure-items, but look instead at the degree of the nyf adjacencies, the equivalence with nyf-extended first difference principle is quite clear. We only need to reverse the decomposition to consider separations/splitting.

Definition 3.2 (Twin-questionable-width). *A twin-questionable-decomposition of an \mathcal{S} -structure S is an $(S, S, k, l+1)$ -mapping-run (a sequence of \mathcal{S} -structures indexed by an ordinal $(S_i)_{i \in l+1}$) such that S_0 is a singleton and $S_l = S$. We always go from S_i to S_{i+1} by a unique separation/splitting of a vertex of S_i in two vertices. The maximum nyf degree of an \mathcal{S} -structure S_i is equal to the supremum of the degrees of its vertices when we only keep the nyf edges. The width of a twin-questionable-decomposition $(S_i)_{i \in l+1}$ is equal to the supremum of the maximum nyf degrees of the S_i . The twin-questionable-width of an \mathcal{S} -structure S , denoted by $\text{twqw}(S)$, is the minimum of the widths of all its twin-questionable-decompositions.*

Theorem 3.3. *The twin-width is equal to the twin-questionable-width.*

4 Merge-width

Definition 4.1 ((Weak) merge-width). *Fix a vertex set and an \mathcal{S} -structure S . A construction sequence is a sequence of steps, maintaining a partition \mathcal{P} of S and a partition of $\binom{S}{2}$ into two sets: resolved “edges” \mathcal{R} with a given adjacency type (it can correspond to edges and non-edges of a graph), and unresolved pairs \mathcal{U} . Initially, \mathcal{P}*

partitions S into singletons, and every pair in $\binom{S}{2}$ is unresolved. In each step, one of two operations is performed:

- merge two parts $X, Y \in \mathcal{P}$, replacing the two parts by their union,
- resolve a pair of parts $X, Y \in \mathcal{P}$ by a given adjacency type (possibly $X = Y$ except in the weak case), declaring all the unresolved pairs $\{a, b\} \in \mathcal{U}$ with $a \in X$ et $b \in Y$ to be of the chosen adjacency type, that is, moving them from \mathcal{U} to \mathcal{R} .

In the end, we require that \mathcal{P} has one part, and that every pair from $\binom{S}{2}$ is resolved with a valid adjacency type. We thus say this is a construction sequence of S . The radius- r width of a construction sequence is the least number k such that at every step in the sequence, the following holds: For every vertex $v \in S$, at most k parts of the current partition \mathcal{P} can be reached from v by a path of length r in the graph of the currently resolved adjacencies. The radius- r merge-width of S , denoted by $\text{mw}_r(S)$ or $\text{wmw}_r(S)$ in the weak case, is the least radius- r width of a construction sequence of S . Finally, a graph class \mathcal{C} has bounded merge-width, resp. bounded weak-merge-width, if $\text{mw}_r(\mathcal{C}) < \infty$, resp. $\text{wmw}_r(\mathcal{C}) < \infty$, for all $r \in \mathbb{N}$, where $\text{mw}_r(\mathcal{C}) = \sup_{S \in \mathcal{C}} \text{mw}_r(S)$, resp. $\text{wmw}_r(\mathcal{C}) = \sup_{S \in \mathcal{C}} \text{wmw}_r(S)$.

This is a naming problem:

- \mathcal{R} for resolved corresponds to \mathcal{A} for alf.
- \mathcal{U} for unresolved corresponds to \mathcal{N} for nyf. Unresolved is just another word for nyf.

Here again, if we do not focus on the size of the structure-items, but look instead the condition on the size of the non-nyf balls, the equivalence with nyf-extended first difference principle for the weak-merge-width is very clear. For the merge-width, we need an extension that is borderline first difference principle:

Definition 4.2 (nyf-pure question). *Given two elements X, Y of a mapping-run, we say that (q, x_q, y_q, S_q) is the question of X, Y , if q is the smallest ordinal such that the adjacency type between x_q and y_q is not nyf. If $x_q = y_q$, we are looking at the adjacency type on a loop ; it can only work for symmetric adjacency types ; in particular, the merge-width of directed graphs is downgraded to weak-merge-width.*

All the other definitions of question can be simulated by this one with the convention that the loops have only the nyf adjacency type. It makes us lose the possibility to consider that the loops on the S_i are only here to fix the loops of S . This is not a problem in the case of merge-width, because we start with all vertices separated; they can thus receive their loops by questionability with S_0 , and after that the loops of the S_i only fix adjacencies between vertices.

Definition 4.3 ((Weak) merge-questionable-width). *Let S be a set of vertices and an S -structure. A merge-questionable-decomposition is an (S, S, p, l) -mapping-run, a sequence of S -structures indexed by an ordinal $(S_i)_{i \in l+1}$ such that $S_0 = S$ and S_l*

is a singleton. We always go from S_i to S_{i+1} by a unique merging/contraction of two vertices of S_i into one.

The radius- r width of a merge-questionable-decomposition of S is the smallest number k such that at each step i , it is true that: For each vertex $v \in S$, at most k vertices of S_i can be reached from v by a path of length at most r in the graph of adjacencies that were fixed until now: by the nyf-extended first difference principle in the weak-case, by the nyf-pure questionability otherwise. The radius- r merge-questionable-width of S , denoted by $\text{mqw}_r(S)$ or $\text{wmqw}_r(S)$ in the weak case, is the smallest radius- r width among the merge-questionable-decompositions of S . Finally, a graph class \mathcal{C} has bounded merge-questionable-width, resp. bounded weak-merge-questionable-width, if $\text{mqw}_r(\mathcal{C}) < \infty$, resp. $\text{wmqw}_r(\mathcal{C}) < \infty$, for all $r \in \mathbb{N}$, where $\text{mqw}_r(\mathcal{C}) = \sup_{S \in \mathcal{C}} \text{mqw}_r(S)$, resp. $\text{wmw}_r(\mathcal{C}) = \sup_{S \in \mathcal{C}} \text{wmqw}_r(S)$.

Theorem 4.4. *The radius- r merge-questionable-width is equal to the radius- r merge-width. Hence, the merge-width is bounded if and only if the merge-questionable-width is bounded.*

Theorem 4.5. *The radius- r weak-merge-questionable-width is equal to the radius- r weak-merge-width. Hence, the weak-merge-width is bounded if and only if the weak-merge-questionable-width is bounded.*

We are almost certain that the graphs of bounded degree do not have bounded weak-merge-width, although they have bounded merge-width. But we don't have yet a formal proof of this conjecture.

The graphs of bounded degree have also a bounded bijective balanced tree-questionable-width (see Lyaudet (2022)); but this is not the case with the twin-width. It would be interesting to know how the twin-width and the (weak) merge-width compare to the distinct variants of tree-questionable-width (see Lyaudet (2025b)).

These translations of widths in the framework of first difference principle or nyf-pure questionability may seem inconsequential. But these translations can be expressed themselves in some logics, and they operate only on sequences of elements, hence on structures of “paths” and not more complicated graphs. It is possible that more general results on the first difference principle or the nyf-pure questionability enable to classify the complexity of computing these various widths, between those that are in P or NC, or those that are NP-hard with or without approximation algorithms, etc.

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