On non-bijective tree-questionable-width

Laurent Lyaudet*

June 30, 2025

Abstract

In this article, we show that there is an infinity of non-bijective tree-questionable-width types, including 8 main types. The study of these types comes from an example of tree-questionable-decomposition of unbounded degree and depth 2 for all binary structure. We hierarchise these types and give links with bijective tree-questionable-width.

Initial version: 2025/06/12 Current version: 2025/06/30

Keywords: first difference principle, binary structures, 2-structures, graphs, tree-questionable-width, hierarchical decompositions

1 Introduction

(Balanced) (bijective or non-bijective) tree-questionable-width was introduced in Lyaudet (2019). We give new non-bijective variants, study the links with the maximum degree of the decomposition tree, and the links with the bijective case. What follows comes from the detailed study of:

Example. Let S be a binary structure of cardinality n; in a non-bijective decomposition, we can repeat each vertex ($n - 1$ times in the finite case), to make "cherries" for each adjacency between two vertices with two leaves, one for each vertex, plus an internal node that links the two leaves/vertices with the correct adjacency type. Then, we link all these "cherries" to a unique root node that collects all the adjacencies.

This example works or not depending on the meaning we give to first difference principle on a tree. Our example dates from the end of 2019 or the beginning of 2020, but the clarification and what follows is more recent; because in a first time, we just had the reflex to focus on the bijective case with bounded degree, as we did with the binary trees in Lyaudet (2022), Lyaudet (2025b) and Lyaudet (2025a).

2 Definitions

Let V be a set of vertices, a (V, k)-*mapping-run* is a sequence of mappings from V to vertices of binary structures of cardinality at most k (the image binary structure is fixed per mapping).

*`https://lyaudet.eu/laurent/`, laurent.lyaudet@gmail.com

It is possible in tree-questionable-decompositions to have nodes of the decompositions that are mapped to an empty mapping-run (of length 0), but the writing gets more complicated. In this article, we will consider that there is always at least one identity mapping on each node. This identity mapping sends each vertex to a unique vertex of a binary structure of cardinality 1; it simplifies the notion of first difference and will help the reader to form her intuition.

We start by technical details on trees intersections.

Definition 2.1 (Common tree, (Tight/Short/Medium/Wide) junction point, (Tight/Short/Medium/Wide) ascending path, Leaf-root path)**.** *Let T be a rooted tree of root r, T_1 be a rooted subtree of T of same root r, and T_2 be a rooted subtree of T of same root r, such that no leaf of T is in T_1 and T_2 at the same time. We call* common tree *of T_1 and T_2, denoted by $\mathrm{CT}(T_1, T_2)$, the tree induced by the intersection of nodes of T_1 and T_2.*

We say that a leaf of $\mathrm{CT}(T_1, T_2)$ is a tight junction point *of $\mathrm{CT}(T_1, T_2)$.*

We say that a node of $\mathrm{CT}(T_1, T_2)$ is a short junction point *of $\mathrm{CT}(T_1, T_2)$ if this node has a son in T_1 that is not in $\mathrm{CT}(T_1, T_2)$, and a son in T_2 that is not in $\mathrm{CT}(T_1, T_2)$.*

We say that a node of $\mathrm{CT}(T_1, T_2)$ is a medium junction point *of $\mathrm{CT}(T_1, T_2)$ if this node has a son in T_1 or a son in T_2 that is not in $\mathrm{CT}(T_1, T_2)$.*

We say that a node of $\mathrm{CT}(T_1, T_2)$ is a wide junction point *of $\mathrm{CT}(T_1, T_2)$ if this node has at least two sons in the tree induced by the union of nodes of T_1 and T_2.*

In a similar way, we define a tight, resp. short, resp. medium, resp. wide, ascending path *as a directed path from a tight, resp. short, resp. medium, resp. wide, junction point of $\mathrm{CT}(T_1, T_2)$ to the root of $\mathrm{CT}(T_1, T_2)$ for some pair $\{T_1, T_2\}$. Hence, a tight ascending path is a* leaf-root path*.*

Lemma 2.2. *A tight junction point is a short junction point. A short junction point is a medium junction point. A medium junction point is a wide junction point.*

Lemma 2.3. *A tight ascending path is a short ascending path. A short ascending path is a medium ascending path. A medium ascending path is a wide ascending path. Hence, a leaf-root path is tight, short, medium, and wide.*

Lemma 2.4. *All (tight, short, medium, or wide) ascending path is included in at least one leaf-root path.*

There are many definitions that generalize rooted trees in the infinite setting, we use this one: We replace the tree by a well-founded order, where the minimal elements/nodes correspond to the leaves; there is a maximum element/node (corresponding to the root); for any two nodes n_1, n_2, the initial sections generated by them, $\mathrm{IS}(n_1), \mathrm{IS}(n_2)$, (subsets of elements/nodes less than n_1, resp. n_2) are either disjoint (disjoint subtrees), or one is contained in the other (indicating that n_1 is an ancestor or a descendant of n_2). As a result, the final section generated by n_1, $\mathrm{FS}(n_1)$, is always a chain/path with a maximum element corresponding to the root. The "inner nodes" are thus the non-minimal elements of this partial order. In the partial sub-order induced by $\mathrm{IS}(n_1) \setminus \{n_1\}$ (that may be without a maximum element), we look at the set of "complete subtrees" maximal for inclusion, that is to say that these are the maximal subsets of nodes $F \subseteq \mathrm{IS}(n_1) \setminus \{n_1\}$ such that $\forall n_3, n_4 \in F$, if n_3 is incomparable with n_4, then $\exists n_5 \in F$ such that $n_3 < n_5$ and $n_4 < n_5$; such a subtree is an initial section generated by a chain; we say this subtree is a "son" of n_1; the cardinality of this set of "sons"/maximal subtrees inclusion-wise defines the degree of n_1. When we reason on some "sons" in the infinite case, since each son is a set of nodes and not a unique node, we consider that it is the "same" son when the two sets of nodes intersect. The reader can verify that in all the reasonings of this article, no "son" according to a first context intersects two "sons" according to a second context ; it validates the junction points definitions and all the rest.

Definition 2.5 (Tree-questionable-decomposition)**.** *Let S be a binary structure. A* (k, α, β)-tree-questionable-decomposition *of S is a triplet (T, ll, nl) (T as tree, ll as leaf labels et nl as node labels) such that:*

- *T is a rooted tree;*
- *the function ll is a surjective mapping (a bijection in the bijective case) from the leaves of T to the vertices of S;*
- *hence, each internal node $node$ is associated to the subset of vertices of S union of the values $ll(l)$ for all leaves l under the node $node$; it defines $ll(node)$;*
- *nl is a mapping with domain the internal nodes of T, such that $nl(node)$ is an $(ll(node), k)$-mapping-run;*
- *as a consequence, each vertex x of S corresponds to a subtree (that is a path in the bijective case) of T, we denote by T_x this subtree, resp. path;*
- *for every pair of vertices $\{x, y\}$, we will look at the first difference principle on $\mathrm{CT}(T_x, T_y)$, according to the distinct junction points;*
- *let P be a tight, short, medium or wide ascending path relatively to $\{x, y\}$; we can define the $(\{x, y\}, k)$-mapping-run obtained by concatenating the $(ll(node), k)$-mapping-runs restricted to $\{x, y\}$ when we take the nodes $node$ of P from the junction point to the root; if a first difference between the images of x and of y exists in this mapping-run, it is the* question *of P; when the path P has a question, it must correspond to two vertices of same adjacency type as between x and y to be valid;*
- *we say that the tree-questionable-decomposition is tight, resp. short, resp. medium, resp. wide, if all the questions of tight, resp. short, resp. medium, resp. wide, ascending paths are valid;*
- *we say that the tree-questionable-decomposition is 1-tight, resp. 1-short, resp. 1-medium, resp. 1-wide, if it is tight, resp. short, resp. medium, resp. wide, and all the tight, resp. short, resp. medium, resp. wide, ascending paths have a valid question;*
- *in the finite case, if $0 < p \leq 1$, we say that the tree-questionable-decomposition is p-tight, resp. p-short, resp. p-medium, resp. p-wide, if it is tight, resp. short, resp. medium, resp. wide, and for all pair of vertices the proportion of tight, resp. short, resp. medium, resp. wide, ascending paths that have a valid question is at least p;*

- *we say that the tree-questionable-decomposition is ϵ-tight, resp. ϵ-short, resp. ϵ-medium, resp. ϵ-wide, if it is tight, resp. short, resp. medium, resp. wide, and for all pair of vertices at least one of the tight, resp. short, resp. medium, resp. wide, ascending paths has a valid question;*
- *α is the depth of the tree T;*
- *β is the depth of the extended tree T' obtained by replacing each internal node by a path of nodes (one for each mapping of the mapping-run associated to the original node).*

k is called the width of the decomposition; α is called the structural depth of the decomposition; β is called the logical depth of the decomposition. The degree of the decomposition is the maximum (or the supremum in the infinite case) of the degrees of the nodes of T.

The original definition given in Lyaudet (2019) corresponds to 1-tight tree-questionable-decompositions.

The introductory example can be reworded:

Lemma 2.6. *Let S be an (infinite) binary structure, it has a $(2, 2, 2)$-tree-questionable-decomposition of unbounded degree that is 1-tight, ϵ-tight, ϵ-short, ϵ-medium, and ϵ-wide.*

3 Comparisons

We obtain all the comparisons for the 8 main types: 1-tight, 1-short, 1-medium, 1-wide, ϵ-tight, ϵ-short, ϵ-medium, and ϵ-wide. We provide some ideas (lemmas and counter-examples) for other cases.

Lemma 3.1. *If a tree-questionable-decomposition is short, then it is tight. If a tree-questionable-decomposition is medium, then it is short. If a tree-questionable-decomposition is wide, then it is medium.*

Lemma 3.2. *If a tree-questionable-decomposition is 1-tight, resp. 1-short, resp. 1-medium, resp. 1-wide, then it is ϵ-tight, resp. ϵ-short, resp. ϵ-medium, resp. ϵ-wide.*

In the finite case, if $0 < p_1 < p_2 \leq 1$, and if a tree-questionable-decomposition is p_2-tight, resp. p_2-short, resp. p_2-medium, resp. p_2-wide, then it is ϵ-tight and p_1-tight, resp. ϵ-short and p_1-short, resp. ϵ-medium and p_1-medium, resp. ϵ-wide and p_1-wide.

Lemma 3.3. *If a tree-questionable-decomposition is 1-short, then it is 1-tight. If a tree-questionable-decomposition is 1-medium, then it is 1-short. If a tree-questionable-decomposition is 1-wide, then it is 1-medium.*

Lemma 3.4. *If a tree-questionable-decomposition is ϵ-short, then it is ϵ-tight. If a tree-questionable-decomposition is ϵ-medium, then it is ϵ-short. If a tree-questionable-decomposition is ϵ-wide, then it is ϵ-medium.*

Proof:

For all pair of vertices, by Lemma 2.4, if there is a short, resp. medium, resp. wide, ascending path with a valid question, then it is included in a leaf-root path. Hence, this leaf-root path has a question, and, by Lemma 3.1, this question is valid. Only the invalid questions separate the ϵ-tight, ϵ-short, ϵ-medium, and ϵ-wide types. Hence, the given inclusions follow from a second application of Lemma 3.1. ■

All the other inclusions for the 8 main types are false; here are counter-examples.

The example decomposition that motivated this article is 1-tight, ϵ-tight, ϵ-short, ϵ-medium, and ϵ-wide, but neither 1-short, nor 1-medium, nor 1-wide.

The following decomposition is of all main types except 1-wide. Consider a binary structure with 2 vertices a, b, such that a is adjacent to b. We make a cherry with two leaves for a and b; and we fix the adjacency between a and b. We duplicate this cherry. We link the roots of these two cherries to the true root of the decomposition, that has just an identity mapping.

The following decomposition is ϵ-tight, ϵ-short, ϵ-medium, and ϵ-wide, but neither 1-tight, nor 1-short, nor 1-medium, nor 1-wide. Consider a binary structure with two vertices a, b, such that a is adjacent to b. We make a cherry with two leaves for a and b; and we fix the adjacency between a and b. We make a second cherry with two leaves for a and b; and we fix nothing (identity mapping). We link the roots of the two cherries to the true root of the decomposition that has also just an identity mapping.

The following decomposition is 1-tight, and ϵ-tight, but neither 1-short, nor ϵ-short, nor 1-medium, nor ϵ-medium, nor 1-wide, nor ϵ-wide. Consider a binary structure with 3 vertices a, b and c, such that a is adjacent to c; and otherwise everything is non-adjacent. We make a cherry with two leaves for a and b; and we fix the non-adjacency between a and b. We link the root of this cherry to the true root of the decomposition. This true root has also a repeated leaf son a, a repeated leaf son b, and a leaf son c. The mapping-run of the true root fixes a adjacent to b and c (but this is covered by the cherry between a and b for the types 1-tight, and ϵ-tight), then fixes b (and a) non-adjacent to c.

The following decomposition is 1-tight, ϵ-tight, 1-short, ϵ-short, but neither 1-medium, nor ϵ-medium, nor 1-wide, nor ϵ-wide. Consider a binary structure with 3 vertices a, b and c, such that a is adjacent to c; and otherwise everything is non-adjacent. We make a cherry with two leaves for a and b; and we fix the non-adjacency between a and b. We link the root of this cherry to the true root of the decomposition. This true root has also a repeated leaf son a and a leaf son c. The mapping-run of the true root fixes a adjacent to b and c (but this is covered by the cherry between a and b for the types 1-tight, ϵ-tight, 1-short, and ϵ-short), then fixes b (and a) non-adjacent to c.

The following decomposition is 1-tight, ϵ-tight, 1-short, ϵ-short, 1-medium, and ϵ-medium, but neither 1-wide, nor ϵ-wide. Consider a binary structure with 3 vertices a, b and c, such that a is adjacent to c; and otherwise everything is non-adjacent. We make a cherry with two leaves for a and b; and we fix the non-adjacency between a and b. We duplicate this cherry and we link the two cherries. We link the union of the two cherries to the true root of the decomposition. This true root has also a leaf son c. The mapping-run of the true root fixes a adjacent to b and c (but this is covered by the cherries between a and b for the types 1-tight, ϵ-tight, 1-short, ϵ-short, 1-medium, and ϵ-medium), then fixes b (and a) non-adjacent to c.

We gave 6 counter-examples for the inclusions. If you take the time to draw the Hasse diagram, and to put the separation lines linked to the counter-examples, you will see that the last 4 counter-examples are sufficient.

Lemma 3.3 has no equivalent in the finite case and with the intermediate proportions of valid questions.

In particular, p-wide or p-medium doesn't imply p-short or p-tight. Just consider for example a decomposition D of a binary structure with two vertices a and b; D has two branches: one deep and one "wide"; on the root of the deep branch there is a valid question. The deep branch is a comb that starts with a cherry for a and b, then adds r leaves with a and adds r internal nodes. The "wide" "branch" contains s cherries for a and b. The last internal node of the deep branch and all the cherries of the "wide" "branch" are linked to the root of D. D is $\frac{1}{s+1}$-tight, $\frac{1}{s+1}$-short, $\frac{r+1}{r+s+1}$-medium, and $\frac{r+1}{r+s+2}$-wide.

To show that p-short doesn't imply p-tight, it is enough to add r leaves with b under the r internal nodes of the previous example. D becomes $\frac{1}{s+1}$-tight, $\frac{r+1}{r+s+1}$-short, $\frac{r+1}{r+s+1}$-medium, and $\frac{r+1}{r+s+2}$-wide.

To see that p-wide doesn't imply p-medium, the example is barely more complicated. Just consider for example a decomposition D of a binary structure with two vertices a and b; D is made of a balanced binary tree of depth l plus a last level made of cherries for a and b; the root of this binary tree has a valid question; this binary tree is then followed toward the root of D by a comb that adds s leaves with a and adds s internal nodes. The last added internal node is the root of D. D is $\frac{2^l}{2^l+s}$-medium and $\frac{2^{l+1}-1}{2^{l+1}-1+s}$-wide. Hence, for $s = 2^{l+1}$, we get about $\frac{1}{3}$-medium and $\frac{1}{2}$-wide. And for $s = 2^{l+h}$, we get $\frac{1}{2^h+1}$-medium, and $\frac{1-\frac{1}{2^{l+1}}}{2^{h-1}+1-\frac{1}{2^{l+1}}}$-wide. Thus the coefficient to go from wide to medium is $\frac{1}{2^h+1} \times \frac{2^{h-1}+1-\frac{1}{2^{l+1}}}{1-\frac{1}{2^{l+1}}} = \frac{2^{h-1}+1-\frac{1}{2^{l+1}}}{2^h+1} \times \frac{1}{1-\frac{1}{2^{l+1}}}$. For all $l \geq 0$, $\frac{2^{h-1}+1-\frac{1}{2^{l+1}}}{2^h+1}$ goes to $\frac{1}{2}$ when we increase h; $\frac{1}{1-\frac{1}{2^{l+1}}}$ goes to 1 when we increase l; hence, we can make it go to $\frac{1}{2}$ by superior values.

This last counter-example can be analysed to show that p-wide implies $\frac{p}{2}$-medium.

Lemma 3.5. *In the finite case, if $0 < p \leq 1$ and if a tree-questionable-decomposition is p-wide, then it is $\frac{p}{2}$-medium, and even $\frac{p}{2-p}$-medium.*

Proof:

> We will show that the counter-example above is optimal. Let us remark first that a counter-example on a binary structure with more than two vertices can be pruned into a counter-example with only two vertices, since the minimum ratio/coefficient must be attained for some pair of vertices. Let us call them a and b. Since it brings nothing to have a mapping that fixes a wrong adjacency type between a et b, we can assume that for all node, we have either a unique identity mapping, or we have a unique mapping that fixes the valid adjacency type for all paths coming from junction points below in the tree. Let vmp be the number of medium

ascending paths with a valid question, wmp be the number of medium ascending paths without question, vwp be the number of wide ascending paths with a valid question, and wwp be the number of wide ascending paths without question. The counter-example is $\frac{vmp}{vmp+wmp}$-medium and $\frac{vwp}{vwp+wwp}$-wide. The goal of the counter-example is to minimise $\frac{vmp\times(vwp+wwp)}{(vmp+wmp)\times vwp}$. Since a medium junction point is also a wide junction point, the medium ascending paths without question cannot be more than the wide ascending paths without question: $wmp \leq wwp$. Hence, the closest wmp gets to wwp, the more the ratio decreases. It shows that the parameter s of our counter-example is optimal, since it meets equality. We can simplify the ratio by $\frac{vmp\times(vwp+wmp)}{(vmp+wmp)\times vwp} = \frac{vmp\times vwp+vmp\times wmp}{vmp\times vwp+wmp\times vwp} = \frac{vmp\times vwp+wmp\times vmp}{vmp\times vwp+wmp\times vwp}$. Likewise, $vmp \leq vwp$, but this time we want to get the biggest possible value of vwp compared to vmp in order to decrease the ratio. Let us look at the structure of the subtrees with a valid question. By minimality of the counter-example, apart from its leaves, all its nodes are in CT. Hence all the internal nodes are wide junction points. And if a node that has only leaves as sons is of degree more than 2, we see immediately that we can delete at least one son. Thus, if the leaves are level 0, the property to be in CT and to have degree two is true at level 1. By induction on the levels, if a node above in the tree has more than two sons, we can separate them to make a cherry whose root is a wide but not medium junction point, and at most one leaf of the cherry is a medium junction point (if the separated node was), the second leaf being always a wide junction point. Hence, it increases vwp by one, but not vmp. This is also the case, if only one of the two sons is a leaf and that we replace it by an a, b cherry. Thus, the "valid branches" of an optimal counter-example are always binary trees ended by a, b cherries, whose only nodes that are medium junction points are those of level 1. If there is more than one such branch, we can move one to "graft" it on another such branch in place of a cherry; it will remove one medium junction point but it may be compensated by a new one after pruning the original attachment point of the branch. Hence, our counter-example with a valid binary tree followed by an invalid comb is optimal. And it is sufficient to see that we never meet the ratio $\frac{vmp}{vwp} > \frac{1}{2}$, but that we can get as close as we want with balanced binary trees. Here is the proof for the bound $\frac{p}{2-p}$: $p = \frac{vwp}{vwp+wwp} \Leftrightarrow \frac{p}{vwp} = \frac{1}{vwp+wwp} \Leftrightarrow \frac{vwp}{p} = vwp + wwp \Leftrightarrow wwp = \frac{vwp}{p} - vwp = \frac{vwp-p\times vwp}{p} = vwp \times \frac{1-p}{p}$; $\frac{vmp}{vmp+wmp} \geq \frac{vmp}{vmp+wwp} \geq \frac{\frac{vwp}{2}}{\frac{vwp}{2}+wwp} = \frac{vwp}{vwp+2wwp} = \frac{vwp}{vwp+2\times(vwp\times\frac{1-p}{p})} = \frac{1}{1+2\times\frac{1-p}{p}} = \frac{p}{p+2\times(1-p)} = \frac{p}{2-p}$. ■

4 Results on the degree

Lemma 4.1. *If a tree-questionable-decomposition is bijective, it is of all types (the 8 main types, and the other types for the finite case).*

Proof:

$\mathrm{CT}(T_x, T_y)$ is a path. There is only one mapping-run, and it has a valid question.

■

Lemma 4.2. *Let S be a finite binary structure. A 1-short, resp. ϵ-short, resp. 1-medium, resp. ϵ-medium, resp. 1-wide, resp. ϵ-wide, (k, α, β)-tree-questionable-decomposition with maximum degree Δ can be converted into a 1-short, resp. ϵ-short, resp. 1-medium, resp. ϵ-medium, resp. 1-wide, resp. ϵ-wide, $(k, \alpha \times \lceil \lg(\Delta) \rceil, \beta \times \lceil \lg(\Delta) \rceil)$-tree-questionable-decomposition with degree two.*

Proof:

We can separate each node of degree more than two into a cherry in a balanced way. We put an identity mapping on the two leaves of the cherry. The questions will thus not be on these leaves.

All the proof hereafter works for all pair of vertices $\{x, y\}$.

We do not create any question, but we create/move junction points toward the leaves only if the separated node was a junction point. Indeed, if a short junction point appears on a leaf of the cherry, then one of its sons is in T_x but not in T_y, and another one is in T_y but not in T_x; since these sons were sons of the root of the cherry, this root was indeed a short junction point. Similarly, if a medium junction point appears on a leaf of the cherry, then at least one of its sons is in T_x but not in T_y (or in T_y but not in T_x); since this son was a son of the root of the cherry, this root was indeed a medium junction point. Last, if a wide junction point appears on a leaf of the cherry, then at least two of its sons are in the union of T_x and T_y; since these sons were sons of the root of the cherry, this root was indeed a wide junction point.

Hence, the new short, resp. medium, resp. wide, ascending paths have a question if and only if there was a question starting from the original short, resp. medium, resp. wide, junction point. Thus, the condition to always have a question on each short, resp. medium, resp. wide, ascending path and that it is valid is fulfilled for 1-short, resp. 1-medium, resp. 1-wide. And the condition to have a question on at least one short, resp. medium, resp. wide, ascending path and that it is valid is fulfilled for ϵ-short, resp. ϵ-medium, resp. ϵ-wide.

■

Corollary 4.3. *Let S be a finite binary structure of cardinality n, it has an ϵ-tight, ϵ-short, ϵ-medium, and ϵ-wide $(2, \lceil 2 \times \lg(n) \rceil + 1, \lceil 2 \times \lg(n) \rceil + 1)$-tree-questionable-decomposition with degree two.*

Corollary 4.4. *Modulo a logarithmic factor on the depth, a bijective tree-questionable-decomposition of a finite binary structure can be converted into a bijective tree-questionable-decomposition, (binary/)with degree two.*

Thus, the degree doesn't matter much when we search for bijective tree-questionable-decompositions that are balanced in a polylogarithmic way.

Lemma 4.5. *In a binary tree, a short junction point is a leaf.*

Corollary 4.6. *If a tree-questionable-decomposition of degree 2 is tight, resp. 1-tight, resp. ϵ-tight, resp. p-tight, then it is short, resp. 1-short, resp. ϵ-short, resp. p-short. If a tree-questionable-decomposition of degree 2 is p-short, then it is p-tight.*

5 Pruning

We start by stating something obvious:

Lemma 5.1. *If a node $node$ of a 1-tight, 1-short, 1-medium, or 1-wide tree-questionable-decomposition has 2 sons $node_1$ and $node_2$, such that $ll(node_1) \subseteq ll(node_2)$, then we can delete all the subtree rooted at $node_1$.*

Lemma 5.2. *If a 1-wide tree-questionable-decomposition has two leaves mapped to the same vertex, we can delete one and maybe merge its parent node $node$ with its second son, if $node$ ends up being of degree 1.*

Proof:

> A leaf pruning doesn't create any wide junction point (but a wide junction point can become a medium junction point; it excludes this result for 1-medium; and a medium junction point can become a short junction point; it excludes this result for 1-short). And since the leaf is redundant, there is still at least one wide junction point for each pair of vertices. Since with 1-wide type, every wide junction point generates a valid question, the property to still have a valid question is fulfilled. ■

Corollary 5.3. *The 1-wide non-bijective tree-questionable-width is equal to the bijective tree-questionable-width.*

6 Conclusion

After this study, only the following tree-questionable-widths are still truly open in the non-bijective case: the 1-tight balanced tree-questionable-width with bounded degree; the 1-short balanced tree-questionable-width; the 1-medium balanced tree-questionable-width. The "ϵ" types are too powerful and decomposes everything, no matter the additional constraints. There also remains the more classical open problem of the bijective tree-questionable-width. Our counter-examples contain a lot of redundant and slighty stupid structures; it is possible that canonisation processes of the decompositions yield equality of certain types after canonisation. The pruning results are a first step in this direction, but it will probably need more complex transformation results. A lot of things are configurable with the tree-questionable-width, particularly in the non-bijective case. We could complexify the problem further: by looking at various bounds on the degree (logarithmic bounds for example); or by looking at bounds on the "surjectivity", like asking that a vertex of a finite binary structure is the image of at most a logarithmic number of leaves. We could even allow invalid questions, and only ask for a majority of valid questions; in that case or for the variant types with proportion, we could study probabilistic sampling, etc.

Thanks God! Thanks Father! Thanks Jesus! Thanks Holy-Spirit!

References

L. Lyaudet. On finite width questionable representations of orders. *CoRR*, abs/1903.02028, 2019. URL http://arxiv.org/abs/1903.02028.

L. Lyaudet. Diviser n'est pas régner ? *preprint*, abs/202205, 2022. URL https://lyaudet.eu/laurent/Publi/Journaux/LL2022DiviserNestPasRegner/LL2022DiviserNestPasRegner_v6.pdf.

L. Lyaudet. On tree-width and tree-questionable-width. *preprint*, abs/202506-2, 2025a. URL https://lyaudet.eu/laurent/Publi/Journaux/LL2025TreewidthTreequestionablewidth/LL2025TreewidthTreequestionablewidth_en_v1.pdf.

L. Lyaudet. Sur la largeur arborescente et la largeur arborescente questionnable. *preprint*, abs/202506-1, 2025b. URL https://lyaudet.eu/laurent/Publi/Journaux/LL2025TreewidthTreequestionablewidth/LL2025TreewidthTreequestionablewidth_fr_v1.pdf.