On tree-width and tree-questionable-width

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Abstract

In this note, we show that graph classes of bounded tree-width have bounded bijective balanced tree-questionable-width.

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1 Introduction

Tree-width was introduced in Robertson and Seymour (1986) and Halin (1976). (Balanced) (bijective) tree-questionable-width was introduced in Lyaudet (2019). We showed in Lyaudet (2019) that if we allow non-bijective tree-questionable-decompositions, then a tree decomposition of a graph can be converted into a non-bijective tree-questionable-decomposition of roughly the same width and depth. This result associated with Bodlaender's theorem (Bodlaender (1988)) which gives a tree decomposition of logarithmic depth gives us a non-bijective balanced tree-questionable-decomposition of bounded width. We asked in Lyaudet (2019): how these two graph invariants compare if we enforce that the tree-questionable-decomposition is bijective. In this note, we publish our result, obtained during the 2020 pandemic, announced at J.G.A. 2023[1] which shows that graph classes of bounded tree-width have bounded bijective balanced tree-questionable-width.

2 Definitions

The tree decomposition/width of a graph are sufficiently known to avoid giving their definition in this note.

Let V be a set of vertices, a (V, k)*-mapping-run* is a sequence of mappings from V to vertices of binary structures of cardinality at most k (the image binary structure is fixed per mapping).

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[1] see `https://jga2023.sciencesconf.org/` and `https://jga2023.sciencesconf.org/data/pages/2_Laurent_Lyaudet.pdf`

Definition 2.1. *Let S be a binary structure. A* bijective (k, α, β)-tree-questionable-decomposition *of S is a triplet (T, ll, nl) (T as tree, ll as leaf labels et nl as node labels):*

- *T is a rooted binairy tree;*
- *leaves of T are in bijection, through the function ll, with vertices of S;*
- *hence, each internal node $node$ is associated to the subset of vertices of S union of the values $ll(l)$ for all leaves l under the node $node$; it defines $ll(node)$;*
- *nl is a mapping with domain the internal nodes of T, such that $nl(node)$ is an $(ll(node), k)$-mapping-run,*
- *as a consequence, each vertex of S corresponds to a sub-tree (that is a path in the bijective case) of T, and since the intersection of two trees, resp. paths, is a tree, resp. path, we also get a path corresponding to all couple of vertices (x, y). Thus, we can define the $(\{x, y\}, k)$-mapping-run obtained by concatenating the $(ll(node), k)$-mapping-runs restricted to $\{x, y\}$, and we enforce that the first difference between the images of x and y in this mapping-run exists and that it corresponds to two vertices of same adjacency type as between x and y;*
- *α is the depth of the tree T;*
- *β is the depth of the extended tree T' obtained by replacing each internal node by a path of nodes (one for each mapping of the mapping-run associated to the original node).*

k is called the width of the decomposition; α is called the structural depth of the decomposition; β is called the logical depth of the decomposition.

Lemma 2.2 (7.7 in Lyaudet (2019))**.** *If a finite binary structure has a tree-decomposition of width k and depth d, it has a non-bijective $(k + 2, d + 1, d)$-tree-questionable decomposition.*

3 Result

Lemma 3.1. *If a finite binary structure with p distinct adjacency types has a binary tree-decomposition of width k and depth d, it has a bijective $(2, \leq d \times (k + 2) + k, \leq p \times (d \times (k + 1) + (k - 1)) + d + 1)$-tree-questionable-decomposition.*

Proof:

> Starting from a rooted binary tree-decomposition of width k, as each vertex x of the binary structure is associated to a sub-tree of the tree-decomposition corresponding to the bags containing it, we can map each vertex to the node $N_{root}(x)$ of the tree-decomposition in this sub-tree that is closest to the root. This node $N_{root}(x)$ is unique, it will guarantee that the tree-questionable-decomposition is bijective. Moreover, since the bags contain at most $k + 1$ vertices, $|\{y; N_{root}(y) = N_{root}(x)| \leq k + 1$.

Then, to a leaf l of the tree-decomposition, we associate a comb of depth k that adds, one after the other, the $k+1$ vertices y such that $N_{root}(y) = l$. To a binary internal node n of the tree-decomposition with sons n_1 and n_2, we associate first a fusion internal node that connects the two combs coming from n_1 and n_2, then we associate a comb of depth $k+1$ that adds, one after the other, the $k+1$ vertices y tels que $N_{root}(y) = n$. Hence, the structural depth is $\leq d \times (k+2) + k$.

On the first internal node of a comb coming from a leaf, we put a mapping-run with a unique mapping to a binary structure of size 2 that fixes the adjacency type between the first two vertices taken from the bag of the leaf. The two vertices of S are sent to the two vertices of the binary structure of size 2. Otherwise, on each internal node of a comb, we put a mapping-run toward at most p binary structures of size 2 to fix the adjacency types of the current vertex with the binary structure constructed below in the decomposition. In each of the p mappings, the vertex currently added x has same image as the already added vertices that have an adjacency type with x distinct from the adjacency type of the current binary structure of size 2; and the already added vertices that have an adjacency type with x equal to the adjacency type of the current binary structure of size 2 are sent on the other vertex of the current binary structure of size 2. Last, on each fusion internal node, we put a mapping-run with a unique mapping to a binary structure of size 2 with the default adjacency type of the tree-decomposition (non-adjacent in the standard case of graphs); the vertices coming from n_1 are sent to the first vertex of the binary structure of size 2; the vertices coming from n_2 are sent to the second vertex of the binary structure of size 2. Hence, the logical depth is $\leq 1 + (k-1) \times p + (k+1) \times d \times p + d = p \times (d \times (k+1) + (k-1)) + d + 1$. ■

Again, from Bodlaender (1988), we deduce:

Corollary 3.2. *If a class of binary structures has bounded tree-width, it has bounded balanced bijective tree-questionable-width.*

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