Which notions of convergence are possible in mathematics ?

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Abstract

The goal of this note is to do an exhaustive study of the possibilities of definitions of convergence of a sequence of functions. Clearly, someone must have done it before. But it is possible that nobody cared to publish it (on the Internet).

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1 Reminders

We start by some extracts of Wikipedia pages (see Wikipedia (2026a), Wikipedia (2026b), Wikipedia (2026c), Wikipedia (2026d)).

Let X be a set, (Y, d) be a metric space, and A be a subset of X. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions defined on X with values in Y, and f be a function defined on X with values in Y.

Pointwise convergence on A:
$\forall x \in A, \forall \varepsilon > 0, \exists N_{\varepsilon,x} \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq N_{\varepsilon,x} \Rightarrow d(f_n(x), f(x)) \leq \varepsilon.$

Uniform convergence on A:
$\forall \varepsilon > 0, \exists N_{\varepsilon} \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq N_{\varepsilon} \Rightarrow (\forall x \in A, d(f_n(x), f(x)) \leq \varepsilon).$

2 What are the other possible definitions?

Simplifying assumption 1: $\forall \varepsilon > 0$
We define the convergence by the goal $d(f_n(x), f(x)) \leq \varepsilon$ that is quantified by the variable ε that we want as small as possible. From a logical viewpoint, we want the formula to be true $\forall \varepsilon > 0$. This is the only translation of a delta as small as we want with the building blocks of logical formalism. This goal

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exclude the definition of convergence according to Cauchy criterion, but we will look into this later.

Simplifying assumption 2: $\forall x \in A$
We want that it converges on all A, hence $\forall x \in A, d(f_n(x), f(x)) \leq \varepsilon$. Indeed, there is no need for $\forall x \notin SetOfZeroMeasure$ for applications to integration for example. It is sufficient to define correctly the subset $A \subseteq X$ before asking about convergence.

Simplifying assumption 3: $\exists m \in \mathbb{N}, n \geq m \Rightarrow$
We want that the convergence goal is always reached from some rank and after, not once in two, not from time to time, not just an infinity of times.
Simplifying assumption 3': $\forall n \in \mathbb{N}$
Without constraint, it would become $\forall n \in \mathbb{N}, \forall x \in A, f_n(x) = f(x)$, since all functions f_n would be infinitely close of $f(x)$, hence equals. In particular, $\forall n \in \mathbb{N}$ becomes implicit. We can always write it just before $n \geq m$.

Hence the possible choices are

$$\text{@Quantifiers1@}, \forall n \in \mathbb{N}, \qquad n \geq m \Rightarrow \text{@Quantifiers2@}, d(f_n(x), f(x)) \leq \varepsilon.$$

Let us remind that

$$\begin{aligned}
&\text{@Quantifiers1@}, \forall n \in \mathbb{N}, n \geq m \Rightarrow \text{@Quantifiers2@}, d(f_n(x), f(x)) \leq \varepsilon \\
\Leftrightarrow\ &\text{@Quantifiers1@}, \forall n \in \mathbb{N}, \neg(n \geq m) \vee (\text{@Quantifiers2@}, d(f_n(x), f(x)) \leq \varepsilon) \\
\Leftrightarrow\ &\text{@Quantifiers1@}, \text{@Quantifiers2@}, \forall n \in \mathbb{N}, \neg(n \geq m) \vee d(f_n(x), f(x)) \leq \varepsilon \\
\Leftrightarrow\ &\text{@Quantifiers1@}, \text{@Quantifiers2@}, \forall n \in \mathbb{N}, n \geq m \Rightarrow d(f_n(x), f(x)) \leq \varepsilon \\
\Leftrightarrow\ &\text{@Quantifiers@}, \forall n \in \mathbb{N}, n \geq m \Rightarrow d(f_n(x), f(x)) \leq \varepsilon.
\end{aligned}$$

The variables with quantifiers are to be chosen among:

- $\forall \varepsilon > 0$,
- $\forall x \in A$,
- $\exists m \in \mathbb{N}$.

And the rules of logic make that chosing $\forall \varepsilon > 0, \forall x \in A$ or $\forall x \in A, \forall \varepsilon > 0$ are equivalent. Hence, @Quantifiers@ has 4 possible choices:

- $\forall \varepsilon > 0, \forall x \in A, \exists m \in \mathbb{N}$,
- $\exists m \in \mathbb{N}, \forall \varepsilon > 0, \forall x \in A$,
- $\forall \varepsilon > 0, \exists m \in \mathbb{N}, \forall x \in A$,
- $\forall x \in A, \exists m \in \mathbb{N}, \forall \varepsilon > 0$.

It yields:

- $\forall \varepsilon > 0, \forall x \in A, \exists m \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq m \Rightarrow d(f_n(x), f(x)) \leq \varepsilon$, this is pointwise convergence,

- $\exists m \in \mathbb{N}, \forall \varepsilon > 0, \forall x \in A, \forall n \in \mathbb{N}, n \geq m \Rightarrow d(f_n(x), f(x)) \leq \varepsilon$, this is uniform supra-convergence, supra-convergence means that from some rank and on, we have bluntly the equality with the limit,

- $\forall \varepsilon > 0, \exists m \in \mathbb{N}, \forall x \in A, \forall n \in \mathbb{N}, n \geq m \Rightarrow d(f_n(x), f(x)) \leq \varepsilon$, this is uniform convergence,

- $\forall x \in A, \exists m \in \mathbb{N}, \forall \varepsilon > 0, \forall n \in \mathbb{N}, n \geq m \Rightarrow d(f_n(x), f(x)) \leq \varepsilon$, this is pointwise supra-convergence.

With our simplifying assumptions, there are only 4 definitions of convergence:

- pointwise convergence (P.C.),
- uniform convergence (U.C.),
- pointwise supra-convergence (P.S.C.),
- uniform supra-convergence (U.S.C.).

(Supra-convergence = equality from some rank and on.)

Definition 2.1 *Let X be a set, (Y, d) be a metric space, and A be a subset of X. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions defined on X with values in Y, and f be a function defined on X with values in Y.*

- *Pointwise Convergence (P.C.): We say the sequence $(f_n)_{n\in\mathbb{N}}$ converges pointwise to f on A if: $\forall \varepsilon > 0, \forall x \in A, \exists m \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq m \Rightarrow d(f_n(x), f(x)) \leq \varepsilon$,*

- *Uniform Convergence (U.C.): We say the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f on A if: $\forall \varepsilon > 0, \exists m \in \mathbb{N}, \forall x \in A, \forall n \in \mathbb{N}, n \geq m \Rightarrow d(f_n(x), f(x)) \leq \varepsilon$,*

- *Pointwise Supra-Convergence (P.S.C.): We say the sequence $(f_n)_{n\in\mathbb{N}}$ supra-converges pointwise to f on A if: $\forall x \in A, \exists m \in \mathbb{N}, \forall \varepsilon > 0, \forall n \in \mathbb{N}, n \geq m \Rightarrow d(f_n(x), f(x)) \leq \varepsilon$,*

- *Uniform Supra-Convergence (U.S.C.): We say the sequence $(f_n)_{n\in\mathbb{N}}$ supra-converges uniformly to f on A if: $\exists m \in \mathbb{N}, \forall \varepsilon > 0, \forall x \in A, \forall n \in \mathbb{N}, n \geq m \Rightarrow d(f_n(x), f(x)) \leq \varepsilon$.*

Theorem 2.2

- *The uniform convergence implies the pointwise convergence.*
- *The uniform supra-convergence implies the pointwise supra-convergence.*
- *The pointwise supra-convergence implies the pointwise convergence.*
- *The uniform supra-convergence implies the uniform convergence.*

We have this diagram of relations between the 4, where the up implies the down.

```
      U.S.C.
     /    \
    /      \
  U.C.     P.S.C.
    \      /
     \    /
      P.C.
```

And we see that U.C. and P.S.C. are independent.

Counter-example 2.3 *Let $f_n(x) = \frac{1}{n}$ and $f(x) = 0$. (Yes, we do not use x, these are constant functions.) We have the uniform convergence of $(f_n)_{n\in\mathbb{N}}$ to f. But we never have $f_n(x) = f(x)$ for no n, hence we don't have pointwise supra-convergence.*

Counter-example 2.4 *$X = A = \mathbb{N}$. Let $f(x)$ be any function. let $f_n(x)$ be such that: $f_x(x) = f(x)$, same for $n \geq x, f_n(x) = f(x)$, $f_y(x) = f(x) - y$ for $y < x$. (Yes again, at the start the distance increases, this is an advantage of counter-examples, we can be creative :).). We do have the pointwise supra-convergence but not the uniform convergence of $(f_n)_{n\in\mathbb{N}}$ to f. Because $\forall\varepsilon > 0$, there is an $m \in \mathbb{N}$, such that $\varepsilon < m$, hence in particular no uniform convergence because of $f(m+1)$.*

We can even see that we have the following result that is slightly less trivial.

Theorem 2.5 *The uniform convergence with pointwise supra-convergence don't imply uniform supra-convergence.*

Counter-example 2.6 *$X = A = \mathbb{N}$. Let $f(x)$ be any function. Let $f_n(x)$ be such that: $f_x(x) = f(x)$, same for $n \geq x, f_n(x) = f(x)$, $f_y(x) = f(x) - 1/y$ for $y < x$. We do have the pointwise supra-convergence and also the uniform convergence of $(f_n)_{n\in\mathbb{N}}$ to f. But we also see that we don't have uniform supra-convergence, because equality is reached further and further away when x increases.*

These counter-examples can give us a false hope that P.S.C. implies U.C., if we add a common sense constraint of always getting closer to the goal:

Definition 2.7 (To get closer) *Let X be a set, (Y, d) be a metric space, and A be a subset of X. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions defined on X with values in Y, and f be a function defined on X with values in Y.*

- *To get closer in the broad sense: We say the sequence $(f_n)_{n\in\mathbb{N}}$ gets closer to f in the broad sense on A if: $\forall x \in A, \forall n_1, n_2 \in \mathbb{N}, n_1 < n_2 \Rightarrow d(f_{n_2}(x), f(x)) \leq d(f_{n_1}(x), f(x))$,*

- *To get closer in the strict sense: We say the sequence $(f_n)_{n\in\mathbb{N}}$ gets closer to f in the strict sense on A if: $\forall x \in A, \forall n_1, n_2 \in \mathbb{N}, n_1 < n_2 \Rightarrow (d(f_{n_2}(x), f(x)) < d(f_{n_1}(x), f(x)) \vee d(f_{n_2}(x), f(x)) = d(f_{n_1}(x), f(x)) = 0)$.*

Simplifying assumption 4:
What if instead of getting closer from the start, we get closer from some rank and on? This is a little like periodic sequences compared to ultimately periodic sequences. In that case, either we add the word ultimately to the definition, with a quantifier to say that it gets closer only when $m < n_1$, or we note that if the sequence is finite then we can keep directly the last function of the sequence, and if the sequence is infinite, it doesn't cost much to remove the first functions of the sequence to go from a sequence of functions that gets closer ultimately to a sequence of functions that gets closer.

We then see that it is not sufficient that the sequence $(f_n)_{n\in\mathbb{N}}$ supra-converges pointwise to f on A and that it gets closer to f on A in order that it also converges uniformly to f.

Counter-example 2.8 *$X = A = \mathbb{N}$. Let $f(x)$ be any function. let $f_n(x)$ be such that: $f_x(x) = f(x)$, same for $n \geq x, f_n(x) = f(x)$, $f_y(x) = f(x) - 1$ for $y < x$. We do have the pointwise supra-convergence and we get closer in the broad sense, but we do not have the uniform convergence of $(f_n)_{n\in\mathbb{N}}$ to f. Because $\forall 0 < \varepsilon < 1$, we have no uniform convergence because of $f_m(m+1)$.*

Counter-example 2.9 *$X = A = \mathbb{N}$. Let $f(x)$ be any function. Let $f_n(x)$ be such that: $f_x(x) = f(x)$, same for $n \geq x, f_n(x) = f(x)$, $f_y(x) = f(x) - 1 - 1/y$ for $y < x$. We do have the pointwise supra-convergence and we get closer in the strict sense, but we do not have the uniform convergence of $(f_n)_{n\in\mathbb{N}}$ to f.*

We can now relax some simplifying assumptions. What happens when we have $\forall \varepsilon \geq 0$? If we also have $d(f_n(x), f(x)) \leq \varepsilon$, we see that the pointwise convergence, resp. uniform convergence, definitions become pointwise supra-convergence, resp. uniform supra-convergence, definitions. If we have instead $\forall \varepsilon \geq 0$ and $d(f_n(x), f(x)) < \varepsilon$, since we cannot have negative distance, we obtain definitions without real case. If we have instead $\forall \varepsilon > 0$ and $d(f_n(x), f(x)) < \varepsilon$, we obtain equivalent definitions.

3 Cauchy's criteria

Remind that Cauchy's criteria say if the sequence of functions converges and how, in a way that is independent of the knowledge of a possible limit.

Let X be a set, (Y, d) be a metric space, and A be a subset of X. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions defined on X with values in Y.

Cauchy's criterion for pointwise convergence on A:
$\forall x \in A, \forall \varepsilon > 0, \exists N_{\varepsilon,x} \in \mathbb{N}, \forall n_1, n_2 \in \mathbb{N}, n_1 \geq N_{\varepsilon,x} \wedge n_2 \geq N_{\varepsilon,x} \Rightarrow d(f_{n_1}(x), f_{n_2}(x)) \leq \varepsilon$.

Cauchy's criterion for uniform convergence on A:
$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}, \forall n_1, n_2 \in \mathbb{N}, n_1 \geq N_\varepsilon \wedge n_2 \geq N_\varepsilon \Rightarrow (\forall x \in A, d(f_{n_1}(x), f_{n_2}(x)) \leq \varepsilon)$.

All the remarks on the simplifying assumptions of the previous section still applies.

Definition 3.1 (Convergences in the sense of Cauchy) *Let X be a set, (Y, d) be a metric space, and A be a subset of X. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions defined on X with values in Y.*

- *Pointwise Convergence (C.P.C.): We say the sequence $(f_n)_{n\in\mathbb{N}}$ pointwise converges on A if: $\forall \varepsilon > 0, \forall x \in A, \exists m \in \mathbb{N}, \forall n_1, n_2 \in \mathbb{N}, n_1 \geq m \wedge n_2 \geq m \Rightarrow d(f_{n_1}(x), f_{n_2}(x)) \leq \varepsilon$,*

- *Uniform Convergence (C.U.C.): We say the sequence $(f_n)_{n\in\mathbb{N}}$ uniformly converges on A if: $\forall \varepsilon > 0, \exists m \in \mathbb{N}, \forall x \in A, \forall n_1, n_2 \in \mathbb{N}, n_1 \geq m \wedge n_2 \geq m \Rightarrow d(f_{n_1}(x), f_{n_2}(x)) \leq \varepsilon$,*

- *Pointwise Supra-Convergence (C.P.S.C.): We say the sequence $(f_n)_{n\in\mathbb{N}}$ pointwise supra-converges on A if: $\forall x \in A, \exists m \in \mathbb{N}, \forall \varepsilon > 0, \forall n_1, n_2 \in \mathbb{N}, n_1 \geq m \wedge n_2 \geq m \Rightarrow d(f_{n_1}(x), f_{n_2}(x)) \leq \varepsilon$,*

- *Uniform Supra-Convergence (C.U.S.C.): We say the sequence $(f_n)_{n\in\mathbb{N}}$ uniformly supra-converges on A if: $\exists m \in \mathbb{N}, \forall \varepsilon > 0, \forall x \in A, \forall n_1, n_2 \in \mathbb{N}, n_1 \geq m \wedge n_2 \geq m \Rightarrow d(f_{n_1}(x), f_{n_2}(x)) \leq \varepsilon$.*

Theorem 3.2 *With the convergences in the sense of Cauchy, we have:*

- *The uniform convergence implies the pointwise convergence.*

- *The uniform supra-convergence implies the pointwise supra-convergence.*

- *The pointwise supra-convergence implies the pointwise convergence.*

- *The uniform supra-convergence implies the uniform convergence.*

Theorem 3.3

- *The pointwise convergence to a function f and the uniform convergence in the sense of Cauchy imply the uniform convergence to f.*

- *The pointwise convergence to a function f and the pointwise supra-convergence in the sense of Cauchy imply the pointwise supra-convergence to f.*

- *The pointwise convergence to a function f and the uniform supra-convergence in the sense of Cauchy imply the uniform supra-convergence to f.*

4 Approximate convergence

We can find inspiration in approximation algorithms to define convergences: a convergence up to an additive constant, or a convergence up to a multiplicative constant. $d(f_n(x), f(x)) \leq \varepsilon + \delta$ gives us one of the variants up to an additive constant. $d(f_n(x), f(x)) \leq \varepsilon + \delta \times f(x)$ gives us one of the variants up to a multiplicative constant. $d(f_n(x), f(x)) \leq \delta \times f(x)$ gives us another of the variants up to a multiplicative constant.

First remark: it doesn't have a lot of meaning to define a convergence up to a multiplicative constant in the sense of Cauchy, because we see that we need to fix the goal.

Second remark: there is a lot of equivalent formulas in the additive case.

$$\begin{aligned} &..., \forall \varepsilon > 0, ..., n \geq m \Rightarrow d(f_n(x), f(x)) \leq \varepsilon + \delta \\ \Leftrightarrow\ &..., \forall \varepsilon > 0, ..., n \geq m \Rightarrow d(f_n(x), f(x)) < \varepsilon + \delta \\ \Leftrightarrow\ &..., \forall \varepsilon > \delta, ..., n \geq m \Rightarrow d(f_n(x), f(x)) \leq \varepsilon \\ \Leftrightarrow\ &..., \forall \varepsilon > \delta, ..., n \geq m \Rightarrow d(f_n(x), f(x)) < \varepsilon. \end{aligned}$$

We have no guarantee to reach the additive approximation constant, but we get as close as we want, and better yet when the quantifiers are those of supra-convergence we reach the additive approximation constant.

$$\begin{aligned} &..., \forall \varepsilon \geq 0, ..., n \geq m \Rightarrow d(f_n(x), f(x)) \leq \varepsilon + \delta \\ \Leftrightarrow\ &..., \forall \varepsilon \geq \delta, ..., n \geq m \Rightarrow d(f_n(x), f(x)) \leq \varepsilon \\ \Leftrightarrow\ &..., n \geq m \Rightarrow d(f_n(x), f(x)) \leq \delta. \end{aligned}$$

We always reach the additive approximation constant.

$$\begin{aligned} &..., \forall \varepsilon \geq 0, ..., n \geq m \Rightarrow d(f_n(x), f(x)) < \varepsilon + \delta \\ \Leftrightarrow\ &..., \forall \varepsilon \geq \delta, ..., n \geq m \Rightarrow d(f_n(x), f(x)) < \varepsilon \\ \Leftrightarrow\ &..., n \geq m \Rightarrow d(f_n(x), f(x)) < \delta. \end{aligned}$$

We do better than reach the additive approximation constant, since we can be closer than that.

We see that we have at least 3 big families of variants. To simplify in the rest of this article, we kept the convention to take ε (strictly) larger than δ.

Definition 4.1 *Let X be a set, (Y, d) be a metric space, and A be a subset of X. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions defined on X with values in Y, and f be a function defined on X with values in Y. Let δ be a strictly positive real.*

- *We say that the sequence $(f_n)_{n\in\mathbb{N}}$ converges pointwise to f on A up to an additive constant δ majored if: $\forall \varepsilon > \delta, \forall x \in A, \exists m \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq m \Rightarrow d(f_n(x), f(x)) \leq \varepsilon \Leftrightarrow \forall \varepsilon > \delta, \forall x \in A, \exists m \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq m \Rightarrow d(f_n(x), f(x)) < \varepsilon$.*

- *We say that the sequence* $(f_n)_{n\in\mathbb{N}}$ *converges pointwise to* f *on* A *up to an additive constant* δ *reached if:* $\forall\varepsilon \geq \delta, \forall x \in A, \exists m \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq m \Rightarrow d(f_n(x), f(x)) \leq \varepsilon \Leftrightarrow \forall x \in A, \exists m \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq m \Rightarrow d(f_n(x), f(x)) \leq \delta$.
  This is equivalent with the case where the sequence $(f_n)_{n\in\mathbb{N}}$ *supra-converges pointwise to* f *on* A *up to an additive constant* δ *majored if:* $\forall x \in A, \exists m \in \mathbb{N}, \forall\varepsilon > \delta, \forall n \in \mathbb{N}, n \geq m \Rightarrow d(f_n(x), f(x)) \leq \varepsilon \Leftrightarrow \forall x \in A, \exists m \in \mathbb{N}, \forall\varepsilon > \delta, \forall n \in \mathbb{N}, n \geq m \Rightarrow d(f_n(x), f(x)) < \varepsilon$.
  This is also equivalent with the case where the sequence $(f_n)_{n\in\mathbb{N}}$ *supra-converges pointwise to* f *on* A *up to an additive constant* δ *reached if:* $\forall x \in A, \exists m \in \mathbb{N}, \forall\varepsilon \geq \delta, \forall n \in \mathbb{N}, n \geq m \Rightarrow d(f_n(x), f(x)) \leq \varepsilon \Leftrightarrow \forall x \in A, \exists m \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq m \Rightarrow d(f_n(x), f(x)) \leq \delta$.

- *We say that the sequence* $(f_n)_{n\in\mathbb{N}}$ *converges pointwise to* f *on* A *up to an additive constant* δ *minored if:* $\forall\varepsilon \geq \delta, \forall x \in A, \exists m \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq m \Rightarrow d(f_n(x), f(x)) < \varepsilon \Leftrightarrow \forall x \in A, \exists m \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq m \Rightarrow d(f_n(x), f(x)) < \delta$.
  This is also equivalent with the case where the sequence $(f_n)_{n\in\mathbb{N}}$ *supra-converges pointwise to* f *on* A *up to an additive constant* δ *minored if:* $\forall x \in A, \exists m \in \mathbb{N}, \forall\varepsilon \geq \delta, \forall n \in \mathbb{N}, n \geq m \Rightarrow d(f_n(x), f(x)) < \varepsilon \Leftrightarrow \forall x \in A, \exists m \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq m \Rightarrow d(f_n(x), f(x)) < \delta$.

- *We say that the sequence* $(f_n)_{n\in\mathbb{N}}$ *converges uniformly to* f *on* A *up to an additive constant* δ *majored if:* $\forall\varepsilon > \delta, \exists m \in \mathbb{N}, \forall x \in A, \forall n \in \mathbb{N}, n \geq m \Rightarrow d(f_n(x), f(x)) \leq \varepsilon \Leftrightarrow \forall\varepsilon > \delta, \forall x \in A, \exists m \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq m \Rightarrow d(f_n(x), f(x)) < \varepsilon$.

- *We say that the sequence* $(f_n)_{n\in\mathbb{N}}$ *converges uniformly to* f *on* A *up to an additive constant* δ *reached if:* $\forall\varepsilon \geq \delta, \exists m \in \mathbb{N}, \forall x \in A, \forall n \in \mathbb{N}, n \geq m \Rightarrow d(f_n(x), f(x)) \leq \varepsilon \Leftrightarrow \exists m \in \mathbb{N}, \forall x \in A, \forall n \in \mathbb{N}, n \geq m \Rightarrow d(f_n(x), f(x)) \leq \delta$.
  This is also equivalent with the case where the sequence $(f_n)_{n\in\mathbb{N}}$ *supra-converges uniformly to* f *on* A *up to an additive constant* δ *majored if:* $\exists m \in \mathbb{N}, \forall\varepsilon > \delta, \forall x \in A, \forall n \in \mathbb{N}, n \geq m \Rightarrow d(f_n(x), f(x)) \leq \varepsilon \Leftrightarrow \exists m \in \mathbb{N}, \forall\varepsilon > \delta, \forall x \in A, \forall n \in \mathbb{N}, n \geq m \Rightarrow d(f_n(x), f(x)) < \varepsilon$.
  This is also equivalent with the case where the sequence $(f_n)_{n\in\mathbb{N}}$ *supra-converges uniformly to* f *on* A *up to an additive constant* δ *reached if:* $\exists m \in \mathbb{N}, \forall\varepsilon \geq \delta, \forall x \in A, \forall n \in \mathbb{N}, n \geq m \Rightarrow d(f_n(x), f(x)) \leq \varepsilon \Leftrightarrow \exists m \in \mathbb{N}, \forall x \in A, \forall n \in \mathbb{N}, n \geq m \Rightarrow d(f_n(x), f(x)) \leq \delta$.

- *We say that the sequence* $(f_n)_{n\in\mathbb{N}}$ *converges uniformly to* f *on* A *up to an additive constant* δ *minored if:* $\forall\varepsilon \geq \delta, \exists m \in \mathbb{N}, \forall x \in A, \forall n \in \mathbb{N}, n \geq m \Rightarrow d(f_n(x), f(x)) < \varepsilon \Leftrightarrow \exists m \in \mathbb{N}, \forall x \in A, \forall n \in \mathbb{N}, n \geq m \Rightarrow d(f_n(x), f(x)) < \delta$.
  This is also equivalent with the case where the sequence $(f_n)_{n\in\mathbb{N}}$ *supra-converges uniformly to* f *on* A *up to an additive constant* δ *minored if:* $\exists m \in \mathbb{N}, \forall\varepsilon \geq \delta, \forall x \in A, \forall n \in \mathbb{N}, n \geq m \Rightarrow d(f_n(x), f(x)) < \varepsilon \Leftrightarrow \exists m \in \mathbb{N}, \forall x \in A, \forall n \in \mathbb{N}, n \geq m \Rightarrow d(f_n(x), f(x)) < \delta$.

It already gives us 6 definitions more (with a lot of equivalences). On a similar model, we could define at most 12 more up to a multiplicative constant δ of type 1, for $d(f_n(x), f(x)) \leq \varepsilon + \delta \times f(x)$. And we could again define at most 12 more up to a multiplicative constant δ of type 2, for $d(f_n(x), f(x)) \leq \delta \times f(x)$.

Given the probability to make a mistake, this is not very rewarding if it isn't done directly in Rocq (previously Coq) or in LEAN.

Thanks God! Thanks Father! Thanks Jesus! Thanks Holy-Spirit!

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